A CLASS OF OPERATORS ON SEQUENCE SPACES AND TRANSFERENCE

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CERTIFICATE

This is to certify that the work embodied in the thesis ``A class of operators on sequence spaces and transference'' by A. Michael Alphonse has been carried out under my supervision and has not been submitted elsewhere for a degree.

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CONTENTS

CHAPTER I

INTRODUCTION

In this thesis we study a class of operators, first on the sequence spaces ℓ^p, and then by "transference" on the space $L^p(X,\mathbb{B},m)$, where $(X,\mathbb{B},m)$ is a probability space equipped with an invertible measure preserving transformation U.

The classical case is that of the discrete ergodic Hilbert transform, which is defined as

$$\tilde{H} f(x) = \lim_{N\to\infty} \sum_{k=-N}^{N\,\prime} \frac{f(U^{-k}x)}{k} \quad .$$

where the prime in the summation means exclusion of the term k=0. In 1955, Cotlar [15] proved the almost everywhere convergence of the above series for $f \in L^p(X)$, $1 \le p < \infty$ and that the operator $\tilde{H}$ is bounded on $L^p(X)$ for $1 < p < \infty$ and is of weak type (1,1).

A direct proof of Cotlar's result was given by Petersen [25] in 1983, using the Calderón-Coifman-Weiss principle of transference ([10],[14]). This proof consists in proving ℓ^p inequalities for the maximal discrete Hilbert transform on sequence spaces, that is, for the operator

$$H^* a(n) = \sup_N \left| \sum_{k=-N}^{N\,\prime} \frac{a(n-k)}{k} \right|, \quad a \in \ell^p, \quad 1 \le p < \infty$$

Then, inequalities for the maximal ergodic Hilbert transform

$$\tilde{H}^* f(x) = \sup_N \left| \sum_{k=-N}^{N,} \frac{f(U^{-k}x)}{k} \right|$$

are obtained by transference.

We have generalized Petersen's result in two ways. First, we define and study a class of operators, called S-operators, which includes the discrete Hilbert transform, and which are analogues of singular integral operators on the real line $\mathbb{R}$. Secondly, we study these operators on the space ℓ^p_B, where B is a Banach space and

$$\ell^p_B = \left\{ a = \{a(n)\}_{n\in Z} : a(n) \in B \text{ and } \sum_{n\in Z} \|a(n)\|^p < \infty \right\}$$

The operators are then transferred to the space

$$L^p_B(X) = \left\{ f: X \to B:\ f \text{ is strongly measurable and } \int_X \|f(x)\|^p dm(x) < \infty \right\}.$$

In Chapter II, we give the notation and standard or known results that are needed in later chapters. For each result we give a suitable reference for the proof.

The S-operators are defined in Chapter III as follows:

<u>Definition</u>: A sequence $\phi = \{\phi(n)\}_{n\in Z}$ is called an S-kernel if there exist constants C_1, $C_2 > 0$ such that

S1. $\sum_{-N}^{N} \phi(k)$ converges as $N \longrightarrow \infty$

S2. $\phi(0) = 0$ and $|\phi(n)| \le C_1/|n|$, $n \ne 0$

S3. $|\phi(n+1) - \phi(n)| \le C_2/n^2$, $n \ne 0$.

We prove that an S-kernel, which by S2, is in ℓ^2, has its Plancherel transform in $L^\infty(\mathbb{T})$, where $\mathbb{T}$ denotes the circle group. It follows that the convolution operator (called an S-operator)

$T_\phi a = \phi * a$ is bounded on ℓ^2. Further, the convolution is defined on ℓ^p, $1 \le p < \infty$ since by S2, $\phi \in \ell^r$ for all $1 < r \le \infty$. We then prove that the maximal S-operator

$$T_\phi^* a(n) = \sup_N \left| \sum_{k=-N}^{N} \phi(k)\ a(n-k) \right|$$

is of weak type (p,p), $1 \le p < \infty$. Then by interpolation, it follows that this operator (and hence T_ϕ) is bounded on ℓ^p, $1 < p < \infty$. We give two proofs of this result. The first is a direct proof for Z. The second proof consists in transferring the result on singular integral operators on $\mathbb{R}$. This proof also works for Banach space valued sequence spaces ℓ^p_B for appropriate Banach spaces B.

If $(X, \mathbb{B}, m)$ is a probability space and U an invertible measure preserving transformation on X, then U gives rise to a group of isometries $f \longrightarrow f \circ U^n$ on $L^p_B(X)$, $1 \le p \le \infty$. For an S-kernel ϕ, we define the ergodic S-operator $\tilde{T}_\phi$ by

$$\tilde{T}_\phi\ f(x) = \sum_{k=-\infty}^{+\infty} \phi(k)\ f(U^{-k}x)$$

We first show that if B is a reflexive Banach space, then this series converges almost everywhere for f in a dense subset of $L^p_B(X)$ for $1 \le p \le 2$. The transference principle works in the Banach space setting and we conclude that : If B is a UMD space (Unconditional martingale difference) with an unconditional basis, then $\tilde{T}_\phi^*$ is a bounded operator on $L^p_B(X)$ for $1 < p < \infty$ and is of weak type $(1,1)$. The class UMD of Banach spaces were discovered by Burkholder, who gave a geometric characterization of these spaces [8]. Further he along with Bourgain ([6],[7]) proved that B is a UMD Banach space if and only

if the Hilbert transform is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

In Chapter IV we study the space of sequences of bounded mean oscillation: $BMO(Z)$.

Definition. A sequence $b = \{b(n)\}$ is said to be in $BMO(Z)$ if $\sup_I \frac{1}{\text{card } I} \sum_{k\in I} |b(k)-b_I| = \|b\|_* < \infty$, where the supremum is taken over all intervals I in Z, and $b_I = \frac{1}{\text{card } I} \sum_{k\in I} b(k)$.

The space $BMO(\mathbb{R})$ is well-known (for example [19] and [32]). Even though the proofs of results about $BMO(\mathbb{R})$ carry over to $BMO(Z)$ without much difficulty, we have included outlines of the proofs so as to keep the exposition self-contained and also because these have not appeared elsewhere in the literature.

If $(X,\mathbb{B},m)$ and U are as stated above, the space $BMO(X)$ consists of those functions in $L^1(X)$ for which the sequences $b_x(n) = b(U^{-n}x)$, defined for almost all x, are in $BMO(Z)$ with $\operatorname{ess.\,sup}_x \|b_x\|_* < \infty$.

These spaces play an important role in Chapter V, where we study the commutator $[\mathbf{b}, T_\phi]$ of the operator of pointwise multiplication by a sequence $\mathbf{b} = \{b(n)\}$ and an S-operator T_ϕ. This is given by

$$[\mathbf{b},\ T_\phi]\ \mathbf{a}(n) = \mathbf{b}(n)\ T_\phi\mathbf{a}(n) - T_\phi(\mathbf{ba})(n)$$

We first show that if H denotes the discrete Hilbert transform and if $[b,H]$ is bounded on ℓ^p for some $1 \le p < \infty$, then $b \in BMO(Z)$. On the other hand, if $b \in BMO(Z)$ and T_ϕ is an S-operator, we show that the maximal commutator

$$[b, T_\phi]^* a(n) = \sup_N \left| \sum_{k=-N}^{N} \phi(k)\,[b(n) - b(n-k)]\,a(n-k) \right|$$

is bounded on ℓ^p, $1 < p < \infty$.

Condition S3 plays a crucial role in this proof. Let ϕ_N denote the truncation of ϕ i.e. $\phi_N(k) = \phi(k)$ if $|k| \le N$ and 0 otherwise. The proof of the maximal commutator inequalities would have been simpler if the ϕ_N's satisfied S3 uniformly in N. However, this is not true even for $\phi(n) = 1/n$. To overcome this difficulty we dominate $[b, T_\phi]$ by a sum of two operators $[b, T_{\nu_N}]$ and $[b, T_{\psi_N}]$, whose kernels ν_N and ψ_N satisfy S3 uniformly. Then we prove the boundedness of the corresponding maximal operators on ℓ^p. For this technique we refer to [29].

Finally, with $(X, \mathbb{B}, m)$ and U as above we study the commutator of the operator of pointwise multiplication by a function b on X and an S-operator $\tilde{T}_\phi$,

$$[b, \tilde{T}_\phi]f(x) = b(x)\,\tilde{T}_\phi f(x) - \tilde{T}_\phi\,(bf)(x), \quad f \in L^p(X).$$

To prove the almost everywhere existence of $[b, \tilde{T}_\phi]f$, for $f \in L^p(X)$ and the boundedness of this operator, we again consider the maximal operator

$$[b, \tilde{T}_\phi]^* f(x) = \sup_N \left| \sum_{k=-N}^{N} \phi(k)\,[b(x) - b(U^{-k}x)]\,f(U^{-k}x) \right| .$$

By a transference argument, we prove that if $b \in BMO(X)$, then $[b, \tilde{T}_\phi]^*$ is bounded on $L^p(X)$ for $1 < p < \infty$. Further the converse holds under the hypothesis that U is ergodic and $[b, \tilde{H}]^*$ is bounded on $L^p(X)$ for some p, $1 < p < \infty$.

We have not been able to prove that the hypothesis of the boundedness of $[b, \tilde{H}]$ on $L^p(X)$ implies that $b \in BMO(X)$.

CHAPTER II

NOTATION AND PRELIMINARIES

In this chapter we specify the notation and state standard and known result which are needed in later chapters. For each such result we have given a suitable reference.

2.1. Throughout Z denotes the discrete group of integers with the counting measure, written as card $(\cdot)$ (cardinality of a set). $\mathbb{T}$ stands for the circle group $\{e^{i\theta} : \theta \in [0,2\pi)\}$ with the Lebesgue measure, and $\mathbb{R}$ is the real line with the Lebesgue measure dx. If A is a subset of $\mathbb{T}$ or of $\mathbb{R}$, $|A|$ denotes the Lebesgue measure of A. $\mathbb{C}$ denotes the field of complex numbers.

If (X,μ) is a σ-finite measure space, $(B,\|\cdot\|)$ a Banach space and $1 \le p \le \infty$, then the function spaces $L^p_B(X)$ are defined as :

For $1 \le p < \infty$,

$$L^p_B(X) = \{f : X \to B, \text{ strongly measurable with } \int_X \|f(x)\|^p \, d\mu(x) < \infty\}$$

and for $p = \infty$,

$$L^\infty_B(X) = \{f : X \to B, \text{ strongly measurable with } \operatorname*{ess.sup}_{x\in X} \|f(x)\| < \infty\}$$

Then $L^p_B(X)$ with the norm

$$\|f\|_p = \left(\int_X \|f(x)\|^p \, d\mu(x)\right)^{1/p}, \qquad 1 \le p < \infty$$

and $$\|f\|_\infty = \operatorname*{ess.sup}_{x\in X} \|f(x)\|$$

is a Banach space. In particular, if $B = \mathbb{C}$, we write $L^p(X)$ for the

space $L^p_{\mathbb{C}}(X)$. Further, if $X = Z$, we use the standard notation ℓ^p and ℓ^p_B, for $L^p_{\mathbb{C}}(Z)$ and $L^p_B(Z)$, respectively.

2.2. By an interval I in Z or in $\mathbb{R}$, we always mean a finite interval.

For $j \geq 2$, and I an interval in Z we define an interval jI as follows :

If $I = [n,n+k-1]$, so that card $I = k$, let

$$jI = \Big[n-(j-1)[k/2] - 1,\ n+k+(j-1)[k/2]\Big]$$

where for a real number x, [x] denotes the largest integer less than or equal to x. Then we have

$$\begin{aligned}\text{card } (jI) &= k+2(j-1)\ [k/2] + 2 \\ &\leq j \text{ card } I+2 \leq (j+2) \text{ card } I\end{aligned}$$

2.3. The following covering lemma is easy to prove ([19], p.25).

__**Covering Lemma.**__ Let A be a finite subset of Z and $\{I_j\}$ a finite collection of intervals such that $A \subset \bigcup_j I_j$. Then there exists a subcollection $\{J_k\}$ of mutually disjoint intervals such that $A \subseteq \bigcup_k 3J_k$.

2.4. __Hardy-Littlewood maximal function.__

Let f be a locally integrable function on $\mathbb{R}$. Then the Hardy-Littlewood maximal function of f is defined as

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|} \int |f(x)|dx$$

where the supremum is taken over all intervals which contain x.

Similarly, if $\mathbf{a}=\{a(n)\}_{n\in Z}$ is a sequence, the Hardy-Littlewood maximal sequence of $\mathbf{a}$ is given by

$$Ma(n) = \sup_{n\in I} \frac{1}{\text{card } I} \sum_{k\in I} |a(k)|$$

where again the supremum is taken over all intervals in Z which contain n.

The following well-known theorem is proved using the covering lemma 2.3 exactly as in the case of $\mathbb{R}$ ([19] or [30]).

Theorem. Let $1 \le p \le \infty$. There exist constants $C_p > 0$ such that

(i) If $1 < p \le \infty$, $\|Ma\|_p \le C_p\|a\|_p \quad \forall\, \mathbf{a} \in \ell^p$

and (ii) $\text{card } \{n : Ma(n) > \lambda\} \le \frac{C_1}{\lambda} \|a\|_1 \quad \forall\, \mathbf{a} \in \ell^1$ and $\forall\, \lambda > 0$

2.5. The operators we study in this thesis are initially defined (in the sense of convergence of series or integrals) on a dense subspace of the L^p space under consideration. We then prove inequalities for suitable maximal operators and conclude the almost everywhere (a.e.) convergence of the defining series or integrals for all elements of the L^p space. This is a consequence of the following theorem :

The Banach Principle ([31] or [20]). Let (X,μ) be a measure space, B a Banach space and $1 \le p < \infty$. Let $\{T_n\}$ be a sequence of operators defined on $L^p_B(X)$. Let

$$T^*f(x) = \sup_{n\ge 1} \|T_nf(x)\| .$$

If there exists a positive decreasing function $C(\lambda)$ on $(0,\infty)$ which tends to zero as $\lambda \to \infty$ such that

$$\mu\{x \in X : T^{*}f(x) > \lambda \, \|f\|_p\} \le C(\lambda)$$

Then the set $\{f \in L_B^p(X) : T_n f(x)$ converges a.e.$\}$ is closed in $L_B^p(X)$.

2.6. Marcinkiewicz Interpolation Theorem

We state below Marcinkiewicz interpolation theorem for Banach space valued functions. The proof for this case uses the interpolation theorem for the scalar case ([31]) as indicated in [2].

Theorem. Let (X,μ) be a measure space, B_1 and B_2 two Banach spaces. Let T be a sublinear operator defined on the space $S_{B_1}(X)$ of B_1-valued simple functions on X taking values in the space $M_{B_2}(X)$ of B_2-valued strongly measurable functions on X. Let $1<p\le\infty$. Suppose there exist constants C_1, $C_r > 0$ such that

(a) $$\mu\left\{x \in X : \|Tf(x)\|_{B_2} > t\right\} \le \frac{C_1}{t} \|f\|_1 \quad \forall\, f \in S_{B_1}(X)$$

(b) $$\mu\left\{x \in X : \|Tf(x)\|_{B_2} > t\right\} \le \frac{C_r}{t^r} \|f\|_r^r \quad \forall\, f \in S_{B_1}(X)$$

(if $r = \infty$, (b) is replaced by $\|Tf\|_\infty \le C\|f\|_\infty$).

Then if $1 < p < r$, there exists a constant $C_p > 0$ such that

$$\|Tf\|_p \le C_p\|f\|_p \quad \forall\, f \in S_{B_1}(X).$$

2.7. The Ergodic Hilbert Transform.

Let (X,μ) be a probability space and U an invertible measure-preserving transformation on X. Then the ergodic Hilbert transform is defined as

$$\tilde{H}f(x) = \lim_{N\to\infty} \sum_{k=-N}^{N}{}' \frac{f(U^k x)}{k}$$

[illegible] Hilbert transform? [illegible]
[illegible] $\sum_{1}^{\infty} \frac{1}{k}$ [illegible] $\forall x$

for $f \in L^p(X)$, $1 \leq p < \infty$. The above series converges a.e. and $\tilde{H}$ is a bounded operator for $1 < p < \infty$ and is weak type (1,1) ([25]). These results can be obtained from the boundedness of the discrete Hilbert transform on sequence spaces.

If $\{U_t\}_{t\in\mathbb{R}}$ is a one-parameter group of measure preserving transformations on X, then the continuous analogue of the ergodic Hilbert transform is given by

$$\tilde{H}f(x) = \lim_{\epsilon\to 0} \int_{\epsilon<|t|<1/\epsilon} \frac{f(U_t x)}{t}\,dt$$

and similar results hold for this operator ([26]). For some generalizations and other related results we refer to the work of Sato [28], Wos [33], Gallardo and Martin-Reyes [21] and Coifman-Weiss ([12], [14]).

2.8 The ergodic Hardy Space $H^1(X)$ and its Dual.

With (X,μ) and U as in 2.7, the ergodic Hardy space $H^1(X)$ is defined as

$$H^1(X) = \{f \in L^1(X) : \tilde{H}f \in L^1(X)\}.$$

In [13] Coifman and Weiss showed that, as in the classical case, $H^1(X)$ has a maximal function characterization. They also proved that the dual of $H^1(X)$ can be identified with the space BMO(X) (Bounded Mean Oscillation) which is defined as

$$BMO(X)=\left\{b \in L^1(X): \operatorname*{ess.sup}_{x\in X} \left[\sup_N \frac{1}{2N+1} \sum_{k=-N}^{N} |b(U^k x)-b_{I_N}(x)|\right]=\|b\|_* < \infty\right\}$$

where $b_{I_N}(x) = \frac{1}{2N+1} \sum_{k=-N}^{N} b(U^k x)$.

In [24] Petersen showed that if X is a Lebesgue space then the ergodic BMO space contains $L^\infty(X)$ as a proper subspace.

2.9. Ergodic Rectangles. As in the classical case, the ergodic Hardy space $H^1(X)$ has an atomic decomposition. We shall not need this result in this thesis, so we do not describe the atoms (see [9]). The ergodic atoms are defined using ergodic rectangles and we need some results about these in Chapter V.

Definition. Let E be a subset of X with positive measure and $k \geq 1$ such that $U^i E \cap U^j E = \phi$ if $i \neq j$ and $-k \leq i, j \leq k$. Then the set $R = \bigcup_{i=-k}^{k} U^i E$ is called an ergodic rectangle of length (2k+1) with base E.

For the following results we refer to [1].

Proposition. Let (X,μ) be a non-atomic probability space, U an invertible measure preserving transformation on X and k a positive integer.

(a) If $F \subseteq X$ is a set of positive measure then there exists a subset $E \subseteq F$ of positive measure such that E is a base of an ergodic rectangle of length 2k+1.

(b) There exists a countable family $\{E_j\}$ of ergodic rectangles of length 2k+1 such that $X = \bigcup_j E_j$.

2.10. The transference principle.

We have used the basic idea of Calderón's method of transference and in each case we give the complete proof for the

sake of self-sufficiency. However, it may be appropriate to give here a more general theorem due to Coifman and Weiss [14].

Let G be a locally compact abelian group and (X,μ) a σ-finite measure space. Suppose R is a uniformly bounded, strongly continuous representation of G acting on $L^p(X)$ i.e. the map $g \to R_g$ is strongly continuous as a mapping from G into the space of bounded operators on $L^p(X)$; $R_{gh} = R_g R_h$ $g,h \in G$ and $\sup_{g\in G} \|R_g\| \leq C<\infty$.

Let $\phi \in L^1(G)$ have compact support and let $N_p(\phi)$ be the norm of the convolution operator

$$T_\phi f = \phi * f , \qquad f \in L^p(G).$$

Then the following theorem holds :

Theorem [14]. Let ϕ be as above. Then the operator

$$\tilde{T}_\phi F(x) = \int_G \phi(g) \, (R_{g^{-1}} F)(x)dg$$

is defined on $L^p(X)$ and is a bounded operator with norm at most $C^2N_p(\phi)$.

For further generalizations and a discussion of the hypothesis in the above theorem, we refer to [4]. In this thesis, we have $G = Z$ and the representation is induced by an invertible measure preserving transformation on X.

2.11. UMD Banach Spaces.

In 1981, Burkholder [8] discovered the class of Banach spaces which have the Unconditionality property for Martingale Differences (UMD). Using probabilistic techniques he proved that: If B is a UMD Banach space then the Hilbert transform on $\mathbb{R}$ (or on $\mathbb{T}$) is a bounded operator on $L^p_B(\mathbb{R})$ for $1 < p < \infty$.

The converse of this result was proved by Bourgain [6].

The UMD spaces also have a geometric characterization given by:

There exists a symmetric, biconvex function $\zeta : B \times B \longrightarrow \mathbb{R}$ such that $\zeta(0,0) > 0$ and $\zeta(x,y) \leq |x+y|$ of $|x| \leq |y|$.

We would like to remark that Hilbert spaces and L^p spaces for $1 < p < \infty$ are in the class UMD, and that every UMD space is reflexive [7].

2.12. Throghout the thesis C, C_p , C'... denote the constants which may change from one line to the next.

CHAPTER III

S-OPERATORS ON SEQUENCE SPACES AND ERGODIC S-OPERATORS

In this chapter we study a class of operators, called S-operators, (definition 3.1) which includes the discrete Hilbert transform. These operators are discrete analogues of the singular integral operators in $\mathbb{R}$ ([30] , [32]). By transference, we then consider the corresponding ergodic S-operators on L^p-spaces of Banach space valued functions on X, for appropriate Banach spaces.

§ 1. S - Operators on sequence spaces.

3.1. Definition. A sequence $\{\phi(n)\}$ is said to be an S-kernel if there exist constants C_1 and $C_2 > 0$ such that

S1. $\sum_{n=-N}^{N} \phi(n)$ converges as $N \longrightarrow \infty$ eg $\phi(n) = \frac{1}{n}$

S2. $\phi(0) = 0$ and $|\phi(n)| \leq C_1/|n|$, $n \neq 0$

S3. $|\phi(n+1) - \phi(n)| \leq C_2/n^2$, $n \neq 0$.

If $\phi = \{\phi(n)\}$ is an S-kernel and $\mathbf{a} = \{a(n)\} \in \ell^p$, $1 \leq p < \infty$, define

$$T_\phi a(n) = \phi * \mathbf{a}\ (n) = \sum_{k \in Z} \phi(n-k)a(k).$$

Since S2 implies that $\phi \in \ell^r$ for all $1 < r \leq \infty$, the above convolution is defined. In fact, the series converges even for all $\mathbf{a} \in \ell^p_B$, $1 \leq p < \infty$, where B is any Banach space. The operator T_ϕ defined above will be called an S-operator.

3.2. Examples of S-kernels.

(i) Let $\phi(n) = 1/n$, $n \neq 0$. Then T_ϕ is the well known discrete Hilbert transform.

(ii) $\phi(n) = 1/n \log|n|$, $n \neq 0$, ± 1.

3.3. Remark. It is not difficult to see that if ϕ is an S-kernel, then the truncations $\phi_N(n) = \phi(n)$ if $|n| \leq N$ and 0 otherwise, satisfy

S3'. $$\sum_{|k|>2|n|} |\phi_N(k-n) - \phi_N(k)| \leq C_2$$

where C_2 does not depend on N. We remark that ϕ_N need not satisfy S3 uniformly in N (As an example, take $\phi(n) = 1/n$).

3.4. The following Proposition and the Plancherel theorem shows that an S-operator T_ϕ is bounded on ℓ^2.

Proposition. If $\phi = \{\phi(n)\}$ is an S-kernel, then $\hat{\phi} \in L^\infty(\mathbb{T})$.

Proof. Since $\|\hat{\phi} - \hat{\phi}_N\|_2 = \|\phi - \phi_N\|_2 \longrightarrow 0$ as $N \longrightarrow \infty$, we see that for some subsequence $\{N_j\}$

$$\hat{\phi}(t) = \lim_{j\to\infty} \sum_{k=-N_j}^{N_j} \phi(k)e^{-ikt} \text{a.e.}$$

Therefore it is enough to prove that

$$\sup_N \|\hat{\phi}_N\|_\infty < \infty.$$

Fix $N \geq 1$ and $t \in \mathbb{T}$. We will choose m depending on N and t to estimate $|\hat{\phi}_N(t)|$.

$$| \hat{\phi}_N(t) | = | \sum_{k=-N}^{N} \phi(k) e^{-ikt} |$$

$$\leq | \sum_{|k|\leq m} \phi(k) e^{-ikt} | + | \sum_{m<|k|\leq N} \phi(k) e^{-ikt} |$$

$$= A_1 + A_2 \text{ , say.}$$

Let $m = \min(N, [\pi/|t|])$, where for a nonnegative real number α, $[\alpha]$ denotes the largest integer less than α. Then

$$A_1 \leq | \sum_{|k|\leq m} \phi(k)(e^{-ikt} - 1)| + | \sum_{|k|\leq m} \phi(k)|$$

$$\leq \sum_{|k|\leq m} |\phi(k)||kt| + C$$

$$\leq C_1(2m+1)|t| + C \leq C$$

(Using S1, S2 and the choice of m).

For estimating A_2, we have $m < N$,

$$A_2 \leq | \sum_{k=m+1}^{N} \phi(k)e^{-ikt} | + | \sum_{k=m+1}^{N} \phi(-k)e^{ikt} |$$

$$= A_2' + A_2'' \text{ say.}$$

$$A_2' = | \sum_{k=m+1}^{N-1} ([\phi(k) - \phi(k+1)] \sum_{j=m+1}^{k} e^{-ijt}) + \phi(N) \sum_{j=m+1}^{N} e^{-ijt}|$$

$$\leq \sum_{k=m+1}^{N-1} |\phi(k) - \phi(k+1)| \; |\operatorname{cosec} t/2| + |\phi(N)| \; |\operatorname{cosec} t/2|$$

$$\leq C|\operatorname{cosec} t/2| \; [\sum_{k=m+1}^{N-1} 1/k^2 + 1/N]$$

$$\leq \frac{C \pi}{|t| \; (m+1)} \leq C$$

since $|\text{cosec } t/2| \le \pi/|t|$ and $m+1 \ge \pi/|t|$ in this case. The estimate for A_2'' is similar.

Hence the proof of the Proposition is complete.

3.5. With Proposition 3.4. and S3', the kernels $\{\phi_N\}$ satisfy the hypothesis of Cor.2.4.5.[16] and so we have the following theorem.

Theorem. Let $\phi = \{\phi(n)\}$ be an S-kernel. Then there exist constants $C_p > 0$ such that

(i) if $1 < p < \infty$, $\|T_\phi a\|_p \le C_p \|a\|_p$, $\forall\, a \in \ell^p$

(ii) card $\{n : |T_\phi a(n)| > \lambda\} \le C_1/\lambda \; \|a\|_1$, $\forall\, a \in \ell^1$ and $\lambda > 0$.

3.6. Inequalities for the maximal operator.

For dealing with the ergodic S-operator (section 3) we will need inequalities for the maximal S-operator, which is defined as follows:

$$T_\phi^* a(j) = \sup_N \Big| \sum_{k=-N}^{N} \phi(k) a(j-k) \Big| .$$

Theorem. Let ϕ be an S-kernel. Then for each p, $1 \le p < \infty$, there exists a constant $C_p > 0$ such that

(i) if $1 < p < \infty$, $\|T_\phi^* a\|_p \le C_p \|a\|_p$, $\forall\, a \in \ell^p$

(ii) card $\{n : T_\phi^* a(n) > \lambda\} \le C_1/\lambda \; \|a\|_1$, $\forall\, a \in \ell^1$ and $\lambda > 0$.

Proof. Using Marcinkiewicz interpolation theorem (2.6) we see that it is enough to prove that

card $\{j : T_\phi^* a(j) > \lambda\} \le C_p/\lambda^p \; \|a\|_p^p$, $\forall\, a \in \ell^p$, $1 \le p < \infty$.

For proving this inequality we first assume that the S-kernel ϕ is decreasing on Z_+ and on Z_- with the usual order on Z. Let $\lambda > 0$ and let

$$A_\lambda = \{j : T_\phi^* a(j) > \lambda\}.$$

Let $A \subseteq A_\lambda$ be a finite set, then for each $j \in A$, choose N_j such that

$$\Big| \sum_{k=j-N_j}^{j+N_j} \phi(j-k)a(k) \Big| > \lambda.$$

Write $I_j = [j-N_j,\ j+N_j]$. Then

$$A \subseteq \{j : \Big| \sum_{k=-\infty}^{\infty} \phi(j-k)a(k) \Big| > \lambda/2\} \cup \{j \in A : \Big| \sum_{k \notin I_j} \phi(j-k)a(k) \Big| > \lambda/2\}$$

$$= A_1 \cup A_2, \quad \text{say.}$$

By Theorem 3.5. we have card $A_1 \le C_p/\lambda^p \, \|a\|_p^p$.

To estimate card A_2 we may assume that $a(k) \ge 0$, $\forall\, k \in Z$, by considering the positive and negative parts of the real and imaginary parts of $a(k)$ separately. Then

$$A_2 \subseteq \{j : \sum_{k \notin I_j} \phi(j-k)a(k) > \lambda/2\} \cup \{j : \sum_{k \notin I_j} \phi(j-k)a(k) < -\lambda/2\}$$

$$= A_2' \cup A_2'' \quad \text{say.}$$

Since we have assumed that ϕ is decreasing on Z_+ and Z_-,

$$\lambda/2 < \sum_{k \notin I_j} \phi(j-k)a(k) \le \sum_{k \notin I_j} \phi(j'-k)a(k)$$

for all $j' \in [j-N_j,\ j] \equiv I_j'$.

(For A_2'' we will have $\sum_{k \notin I_j} \phi(j'-k)a(k) < -\lambda/2$ for all $j' \in [j, j+N_j]$).

Since $\{I_j\}_{j \in A_2'}$ cover A_2', therefore by the covering lemma (2.3) we can find a subclass of intervals $\{I_{j_m}\}$ such that these are mutually disjoint and $A_2' \subseteq \bigcup_m 3I_{j_m}$. Then, using (2.2)

$$\begin{aligned}
\text{card } A_2' &\leq \sum_m \text{card } 3I_{j_m} \\
&\leq 5 \sum_m \text{card } I_{j_m} \\
&\leq 10 \sum_m \text{card } I'_{j_m} \\
&\leq 10 \text{ card } [\bigcup_m \{j \in I'_{j_m} : \sum_{k \notin I_{j_m}} \phi(j-k)a(k) > \lambda/2\}] \\
&\leq 10 \text{ card } [\bigcup_m \{ j \in I'_{j_m} : |T_\phi a(j)| > \lambda/4 \}] \\
&\quad + 10 \text{ card } [\bigcup_m \{ j \in I'_{j_m} : | \sum_{k \in I_{j_m}} \phi(j-k)a(k)| > \lambda/4\}] \\
&\leq C_p/\lambda^p \|a\|_p^p + C_p/\lambda^p \sum_m (\sum_{k \in I_{j_m}} |a(k)|^p) \\
&\leq C_p/\lambda^p \|a\|_p^p
\end{aligned}$$

since the intervals $\{I_{j_m}\}$ are disjoint. The estimate for card A_2'' is similar.

We end the proof by showing that any S-kernel ϕ can be written as the difference of two S-kernels, each of which is decreasing on Z_+ and on Z_-.

Define $b(0) = 0$ and

$$b(j) = \phi(j) + \sum_{n \geq j} |\phi(n+1) - \phi(n)| + \sum_{n \leq -j} |\phi(n) - \phi(n-1)|, \text{ if } j > 0$$

and $$= \phi(j) - \sum_{n \leq j} |\phi(n) - \phi(n-1)| - \sum_{n \geq -j} |\phi(n+1) - \phi(n)|, \text{ if } j < 0$$

and let $c(j) = b(j) - \phi(j)$.

First let us verify that $\{b(j)\}$ satisfies S1, S2, and S3 and that it is decreasing on Z_+ and on Z_-.

$\{b(j)\}$ satisfies S1, because $\sum_{j=-N}^{N} \phi(j) = \sum_{j=-N}^{N} b(j)$, $\forall$ N. Since

$$|b(j)| \leq |\phi(j)| + C \sum_{|n| \geq |j|} 1/n^2 \leq C/|j|, \; j \neq 0,$$

$\{b(j)\}$ satisfies S2.

Finally for each $j \neq 0, -1$, we have

$$b(j) - b(j+1) = \phi(j) - \phi(j+1) + |\phi(j+1) - \phi(j)| + |\phi(-j) - \phi(-j-1)|.$$

Hence it follows that $\{b(j)\}$ is decreasing on Z_+ and on Z_- and satisfies S3. The above verification also shows that $\{c(j)\}$ is decreasing and satisfies S1, S2 and S3. This completes the proof.

3.7. We now give another proof of Theorem 3.6., which has the advantage that it works for Banach space valued sequence spaces also. In this proof we transfer the result from $\mathbb{R}$. For this transference in the particular case of Hilbert transform we refer to [22] or [3].

3.8. Definition. A locally integrable function defined on $\mathbb{R}\setminus\{0\}$ is said to be a singular integral kernel on $\mathbb{R}$ if there exist constants C and $C' > 0$ such that

K1. $\displaystyle\int_{\varepsilon<|x|<1/\varepsilon} K(x)dx$ converges as $\varepsilon \longrightarrow 0$

K2. $|K(x)| \le C/|x|$, $x\neq 0$

K3. $|K(x-y) - K(x)| \le C'|y|/x^2$ for $|x|>2|y|$.

We define the maximal singular integral operator as

$$T_K^*f(x) = \sup_{\varepsilon>0} \Big|\int_{\varepsilon<|x-y|<1/\varepsilon} K(x-y)f(y)dy\Big|.$$

The following theorem is well known.

3.9. Theorem. Let K be a singular integral kernel on $\mathbb{R}$. Then for each p, $1 \le p < \infty$, there exists a constant $C_p > 0$ such that

(i) if $1 < p < \infty$, $\|T_K^*f\|_p \le C_p\|f\|_p$, $\forall f \in L^p(\mathbb{R})$

(ii) $|\{x : T_K^*f(x) > \lambda\}| \le C_1/\lambda \; \|f\|_1$, $\forall f \in L^1(\mathbb{R})$ and $\lambda > 0$.

For the proof of Theorem 3.9 see [30] or [31].

3.10. Now we give the second proof of Theorem 3.6. Before that we need to prove a lemma.

3.11. Lemma. Let ϕ be an S-kernel. Let K be the linear extension of ϕ defined as

$$K(x) = (1-t)\phi(j) + t\phi(j+1) \text{ if } x = j + t,\ 0 \le t \le 1.$$

Then K is a singular integral kernel on $\mathbb{R}$.

Proof. For $x \in \mathbb{R}$, we write $[x]$ for the largest integer less than x. Let $x = [x] + t$, where $0 < t \leq 1$. Note that for $x \neq 0$,

$$|K(x)| = |(1-t)\,\phi([x]) + t\,\phi([x]+1)|$$

$$\leq c/(|[x]| + 1) \leq c/|x|.$$

Hence K satisfies **K2**.

Since $\phi(0) = 0$, $\lim_{\varepsilon \to 0} \int_{\epsilon}^{1} K(x)dx$ exists. And

$$\int_{-R}^{R} K(x)dx = \phi([R])/2+\phi(-[R])/2 + \sum_{k=-([R]-1)}^{([R]-1)} \phi(k) + \int_{[R]<|x|\leq R} K(x)dx.$$

But $\phi([R])/2+\phi(-[R])/2 + \int_{[R]<|x|\leq R} K(x)dx \leq c/[R] \longrightarrow 0$ as $R \longrightarrow \infty$,

Therefore, by **S1**, **S2** and **K2**, we have **K1**.

For **K3**, let $x, y \in \mathbb{R}$ such that $|x| > 2|y|$. Let $x = n + t$, $y = j + s$, where $0 \leq t,\ s \leq 1$ and $n,\ j \in Z$.

We consider two cases, namely, $|y| < 1$ and $|y| \geq 1$.

For the first case there are two possibilities namely:

(1)$y \geq 0$, (2)$y < 0$.

Since (1) and (2) are dealt in a similar fashion we only discuss (1). Now $x-y = n+(t-y)$.Let us assume $t-y \geq 0$. (If $t-y < 0$ we write $x-y = (n-1) + [1-(y-t)]$ and proceed.) Now

$$|K(x-y)-K(x)| = |[1-(t-y)]\phi(n) + (t-y)\phi(n+1) - (1-t)\phi(n)-t\phi(n+1)|$$

$$\leq |y|\ |\phi(n) - \phi(n+1)|$$

$$\leq C|y| \min\{1/n^2,\ 1/(n+1)^2\}$$

$$\leq C|y|/x^2.$$

For the case $|y|\geq 1$, first we assume $t-s \geq 0$. (If $t-s < 0$, we write $x-y = (n-j-1) + [1-(s-t)]$ and proceed.) Then

$$|K(x-y) - K(x)| \leq |K(x-y) - K(n-j)| + |\phi(n-j)-\phi(n)| + |K(n)-K(x)|.$$

By the mean value theorem and the fact that the slope of the line joining the points $\phi(k)$ and $\phi(k+1)$ is less than or equal to C/k^2 (by S3) we have

$$\begin{aligned}|K(x-y)-K(x)| &\leq C(t-s)/(n-j)^2 + C|j|/(n-j)^2 + Ct/n^2 \\ &\leq C|y|/(n-j)^2 + C|y|/n^2 \text{ (since } |y|>1) \\ &\leq C|y|/(x-y)^2 + C|y|/x^2 \\ &\leq C|y|/x^2 \text{ (since } |x|>2|y|).\end{aligned}$$

This completes the proof of the lemma.

Second proof of theorem 3.6 Let $\phi = \{\phi(n)\}$ be an S-kernel and K the linear extension of ϕ defined on $\mathbb{R}$. By lemma 3.11, K is a singular integral kernel on $\mathbb{R}$ so that (i) and (ii) of theorem 3.9. hold. Let $\mathbf{a}=\{a(k)\} \in \ell^p$ and define

$$f(x) = \sum_{k=-\infty}^{\infty} a(k)\chi_{I_k}(x),$$

where $I_k=[k-1/4, k+1/4]$ and χ_I denotes the characteristic function of I. Then

$$\begin{aligned}\|f\|_p^p &= \sum_k |a(k)|^p|I_k| \\ &= 1/2 \ \|\mathbf{a}\|_p^p.\end{aligned}$$

Fix $n \geq 0$, then for $j \in Z$ and $x \in I_j$, we have

$$\sum_{|k-j|>n} \phi(j-k)a(k) \quad - \quad 2\int_{|x-y|>n+1/2} K(x-y)f(y)dy$$

$$= \quad 2\sum_{|k-j|>n} \int_{I_k} [\phi(j-k) - K(x-y)]f(y)dy.$$

Now if $y \in I_k$ with $k \neq j$, we see that

$$|x - y - j + k| \leq 1/2 \leq 1/2\ |j - k|,$$

so that by **K3**,

$$|\phi(j-k) - K(x-y)| \leq \frac{C'|(x-y) - (j-k)|}{|j-k|^2}$$

$$\leq \frac{C}{(x-y)^2}.$$

Therefore,

$$\Big|\sum_{|k-j|>n} \phi(j-k)a(k) - \quad 2\int_{|x-y|>n+1/2} K(x-y)f(y)dy\Big|$$

$$\leq C \sum_{|k-j|>n} \int_{I_k} \frac{|f(y)|}{(x-y)^2}\, dy$$

$$\leq C \int_{|x-y|>1/2} \frac{|f(y)|}{(x-y)^2}\, dy$$

$$= C \sum_{k=0}^{\infty} \int_{2^k \geq |x-y| > 2^{k-1}} \frac{|f(y)|}{(x-y)^2}\, dy$$

$$\leq C \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\frac{1}{2^k} \int_{|x-y| \leq 2^k} |f(y)|dy\right)$$

$$\leq C\, Mf(x),$$

where M is the Hardy-Littlewood maximal operator.

Hence $\Big|\sum_{|k-j|>n} \phi(j-k)a(k)\Big| \leq C\,(T_K^* f(x) + Mf(x))$ for $x \in I_j$.

In particular putting n=0,

$$|T_\phi a(j)| \le C\ (T_K^* f(x) + Mf(x)), \text{ for } x \in I_j.$$

Then
$$\begin{aligned} T_\phi^* a(j) &= \sup_n \Big| \sum_{|k-j|\le n} \phi(j-k)a(k) \Big| \\ &\le |T_\phi a(j)| + \sup_n \Big| \sum_{|k-j|>n} \phi(j-k)a(k) \Big| \\ &\le C\ (T_K^* f(x) + Mf(x)), \text{ for } x \in I_j. \end{aligned}$$

Finally if $1 < p < \infty$, we have

$$\begin{aligned} \Big| \sum_{j=-\infty}^{\infty} (T_\phi^* a(j))^p \Big|^{1/p} &\le C\ [\sum_{j=-\infty}^{\infty} \int_{j-1/4}^{j+1/4} (T_K^* f(x) + Mf(x))^p dx\ ]^{1/p} \\ &\le C\ [\int_{\mathbb{R}} (T_K^* f(x))^p dx]^{1/p} + C\ [\int_{\mathbb{R}} (Mf(x))^p dx]^{1/p} \\ &\le C\ \|f\|_p \le C\ \|a\|_p . \end{aligned}$$

The weak (1,1) inequality is proved as follows.

Let $\lambda > 0$. then

$$\begin{aligned} \text{card}\{j : T_\phi^* a(j) > \lambda\} &\le 2|\{x : T_K^* f(x) + Mf(x) > \lambda/C\}| \\ &\le 2|\{x : T_K^* f(x) > \lambda/2C\}| + 2|\{x : Mf(x) > \lambda/2C\}| \\ &\le C/\lambda\ \|f\|_1 \le C/\lambda\ \|a\|_1 . \end{aligned}$$

This completes the proof.

§2. S-Operators on Banach space valued sequence spaces.

As remarked earlier, the boundedness of T_ϕ^* on the Banach space valued sequence spaces ℓ_B^p can be obtained , using Prop. 3.6, from the corresponding results about T_K^* on L_B^p ($\mathbb{R}$). These results are given below (see also 2.11).

3.12. The following theorem essentially is due to Bourgain.

Theorem [5]. Let B be a UMD Banach space with a normalized unconditional basis $\{e_j\}$ and T_K a singular integral operator on $\mathbb{R}$, with kernel K. Let $\bar{T}_K$ be the operator defined as

$\bar{T}_K(f) = \sum_j T_K(f_j)e_j$, where $f = f_j e_j$, $f_j \in L^p(\mathbb{R})$.

Then (i) If $1<p<\infty$, $\|\bar{T}_K f\|_p \le C_p \|f\|_p$, $\forall\ f \in L^p_B(\mathbb{R})$.

(ii) $|\{x\in \mathbb{R} : \|\bar{T}_K f(x)\|>\lambda \}| \le C_1/\lambda \ \|f\|_1$, $\forall\ f \in L^1_B(\mathbb{R})$ and $\lambda > 0$.

For the proof of (i) see [5] . If K is a singular integral kernel and (i) holds for some p, $1< p < \infty$, then (ii) follows, as shown in [2] or [27].

3.15. If B is a Banach space with norm $\|.\|$, and K a singular integral kernel on $\mathbb{R}$, we consider the maximal singular integral operator as

$$T^*_K f(x) = \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} K(x-y)f(y)dy \right\|,$$

where f is a locally integrable B-valued function on $\mathbb{R}$.

To prove the L^p inequalities for T^*_K , we state the following Proposition.

Proposition [32]. Let B be a Banach space with norm $\|.\|$ and K a singular integral kernel on $\mathbb{R}$. Then for $0 < \eta < 1$, the maximal singular integral operator satisfies

$$T^*_K f(x) \le C\ [(M(\|T_K f\|^{\eta})(x))^{1/\eta} + M(\|f\|)(x)],$$

for every compactly supported function f in $L^\infty_B(\mathbb{R})$, where $C = C_\eta$ is independent of f and x.

For the proof of the above Proposition for the scalar valued case can be found in [32]P:291.The same proof works for Banach space valued case also, by replacing $|.|$ by $\|.\|$, wherever necessary.

From Theorem 3.14 and Prop. 3.15 , it is easy to conclude the following :

3.16. Theorem. Let B be a UMD Banach space with an unconditional basis.Then the maximal singular integral operator T_K^* is bounded on $L_B^p(\mathbb{R})$, $1 < p < \infty$, and satisfies the weak $(1\ ,\ 1)$ inequality.

3.17. As we remarked in § 2, the second proof of Theorem 3.6 works for Banach space valued functions also, by replacing $|.|$ by $\|.\|$ wherever necessary. So by Theorem 3.16. we have the following result.

Theorem. Let B be a UMD Banach space with an unconditional basis. Then the maximal S-operator T_ϕ^* is bounded on ℓ_B^p, $1 < p < \infty$, and satisfies the weak $(1,1)$ inequality.

§3. Ergodic S-operators

3.18. Let $(X,\mathbb{B},m)$ be a probability space and U an invertible measure preserving transformation on X. If $(B, \|.\|)$ is a Banach space, we write $L_B^p(X)$ for the space of strongly measurable B-valued functions on X such that

$$\int_X \|f(x)\|^p dm \ < \infty.$$

On $L_B^p(X)$,with respect to $\{U^n\}_{n\in Z}$, we will define and study a class of operators called ergodic S-operators. If $\phi = \{\phi(n)\}$ is an

S-kernel, we define the operator $\tilde{T}_\phi$, a priori, as

$$\tilde{T}_\phi f(x) = \sum_{j\in Z} \phi(j)f(U^{-j}x).$$

We show below that if B is reflexive, then this operator is well defined on a dense subspace of $L^p_B(X)$, $1\leq p\leq 2$.

3.19. Lemma. Let B be a reflexive Banach space and U an invertible measure preserving transformation on a probability space $(X,\mathbb{B},m)$. Let $1 \leq p \leq 2$.

Let $D = \{f\in L^p_B(X): f = g - g\circ U,\ g \in L^\infty_B(X)\} \cup \{f \in L^p_B(X): f=f\circ U \text{ a.e.}\}$

Then D is dense in $L^p_B(X)$.

Proof. Since B is reflexive, $[L^p_B(X)]^* = L^{p'}_{B^*}(X)$, where $1/p + 1/p' = 1$ and B^* is the dual of B , [16]. Now suppose $h^*\in L^{p'}_{B^*}(X)$ and $\langle f,h^*\rangle = 0$, $\forall\ f \in D$.

First, if $f = g - g\circ U$ with $g \in L^\infty_B(X)$, we have

$$\begin{aligned} 0 &= \langle f , h^*\rangle \\ &= \int_X \langle g(x)-g(Ux) , h^*(x)\rangle dm \\ &= \int_X \langle g(x) , h^*(x)-h^*(U^{-1}x)\rangle dm. \end{aligned}$$

Since this holds for all $g \in L^\infty_B(X)$, a dense subset of $L^p_B(X)$, we conclude $h^*(x)-h^*(U^{-1}x) = 0$ a.e. Now note that h^* is almost separably valued, i.e $M=\overline{(\text{ess range } h^*)}$ is separable and reflexive. Therefore M^* is also separable. Let $\{\tilde{b}_j\}$ be a countable dense

subset of M^* and take

$f_j(x) = b_j \overline{\langle b_j, h^*(x)\rangle}$. Then $f_j \in L^p_B(X)$ since $h^* \in L^{p'}_{B^*}(X)$ and $1 \le p \le 2$. Also $f_j = f_j oU$ since $h^* = h^* o\, U$ so that $f_j \in D$ and $\langle f_j, h^*\rangle = 0$. But this implies $\int_X |\langle b_j, h^*(x)\rangle|^2 dm = 0$.

Hence $\langle b_j, h^*(x)\rangle = 0$ a.e. But then $h^*(x) = 0$ a.e. It follows that D is dense in $L^p_B(X)$.

3.20. Lemma. Let $\phi = \{\phi(n)\}$ be an S-kernel. Then $\sum_{k=-N}^{N} \phi(k) f(U^{-k}x)$ converges a.e. for all $f \in D$.

Proof. If $f = foU$, this is obvious by **S1**.

Let $f = g - goU$ with $g \in L^\infty_B(X)$. Then using a partial summation formula we get

$$\left\| \sum_{N \le |k| \le M} \phi(k) f(U^{-k}x) \right\| = \left\| \sum_{N \le |k| \le M} \phi(k) \{ g(U^{-k}x) - g(U^{-k+1}x) \} \right\|$$

$$\le \|g\|_\infty (|\phi(N)| + |\phi(-N)| + |\phi(M)| + |\phi(-M)|)$$

$$+ \|g\|_\infty \left(\sum_{k=N}^{M-1} |\phi(k) - \phi(k+1)| + \sum_{k=-M}^{-N-1} |\phi(k) - \phi(k+1)| \right)$$

$$\le \|g\|_\infty \left(C/N + C \sum_{k=N}^{M} 1/k^2 \right)$$

$$\longrightarrow 0 \text{ as } N, M \longrightarrow \infty.$$

This completes the proof of the lemma.

3.21. Theorem. Let $1 \le p < \infty$. Let B be a Banach space such that the maximal S-operator T_ϕ^* satisfies the following. For each p, $1 \le p < \infty$, there exists a constant $C_p > 0$ such that

(i) $\|T_\phi^* \mathbf{a}\|_p \le C_p \|\mathbf{a}\|_p$, $1 < p < \infty$ and $\mathbf{a} \in \ell_B^p$

(ii) $\operatorname{card}\{ n : T_\phi^* a(n) > \lambda \} \le C_1/\lambda \; \|\mathbf{a}\|_1$, $\forall\, \mathbf{a} \in \ell^1$ and $\lambda > 0$.

Then the maximal ergodic S-operator $\tilde{T}_\phi^*$ defined as

$$\tilde{T}_\phi^* f(x) = \sup_N \Big\| \sum_{k=-N}^{N} \phi(k) f(U^{-k}x) \Big\|$$

is bounded on $L_B^p(X)$, $1 < p < \infty$, and satisfies the weak (1,1) inequality.

Proof. By Marcinkiewicz interpolation theorem(2.6), it is enough to prove $\tilde{T}_\phi^*$ satisfies a weak (p,p) inequality, $1 \le p < \infty$. Fix $N \in Z_+$ and let

$$\tilde{T}_{\phi,N}^* f(x) = \sup_{1 \le n \le N} \Big\| \sum_{k=-n}^{n} \phi(k) f(U^{-k}x) \Big\|.$$

Since $\tilde{T}_\phi^* f(x) = \lim_{N\, \infty} \tilde{T}_{\phi,N}^* f(x)$, it is enough to prove

$$m\{x \in X : \tilde{T}_{\phi,N}^* f(x) > \lambda\} \le C/\lambda^p \; \|f\|_p^p, \;\forall\, \lambda > 0,$$

where the constant C is independent of N. Let M be a positive integer and define

$$a_x^M(k) = \begin{cases} f(U^k x) & \text{if } |k| \le M + N \\ 0 & \text{otherwise} \end{cases}$$

Let $E = \{x \in X : \tilde{T}^{*}_{\phi,N} f(x) > \lambda\}$.

Then

$$m(E) = 1/2M+1 \sum_{j=-M}^{M} m(U^{-j}E)$$

$$= 1/2M+1 \sum_{j=-M}^{M} m\ \{x \in X : \tilde{T}^{*}_{\phi,N} f(U^{j}x) > \lambda\}$$

$$m(E) = 1/2M+1 \sum_{j=-M}^{M} m\{x \in X : T^{*}_{\phi,N} a^{M}_{x}(j) > \lambda\}$$

$$\leq 1/2M+1\ (m \times \text{card})\ \{(x,j) : T^{*}_{\phi,N} a^{M}_{x}(j) > \lambda\}$$

$$= 1/2M+1 \int_{X} \text{card}\{j : T^{*}_{\phi,N} a^{M}_{x}(j) > \lambda\} dm$$

$$\leq 1/2M+1 \int_{X} C/\lambda^{p} \sum_{j=-\infty}^{\infty} \|a^{M}_{x}(j)\|^{p} dm$$

$$= 1/2M+1 \int_{X} C/\lambda^{p} \sum_{j=-(N+M)}^{(N+M)} \|f(U^{j}x)\|^{p} dm$$

$$= C/\lambda^{p}\ [2(N+M)+1]/2M+1\ \|f\|_{p}^{p}$$

By choosing M sufficiently large we get

$$m(E) \leq C/\lambda^{p} \|f\|_{p}^{p}.$$

This completes the proof.

3.22. As we saw in 3.17, if B is a UMD Banach space with an unconditional basis, then the maximal S-operator T^{*}_{ϕ} is bounded on ℓ^{p}_{B}, $1 < p < \infty$, and satisfies the weak (1 , 1) inequality. For such Banach spaces, $\tilde{T}_{\phi}$ is well defined on a dense subset of $L^{p}_{B}(X)$ by lemmas 3.19 and 3.20. Theorem 3.21 and the Banach Principle (2.5) shows that the series defining $\tilde{T}_{\phi} f$ converges a.e. for all $f \in L^{p}_{B}(X)$, $1 \leq p < \infty$. what-can be said about convergence ...

CHAPTER IV

SEQUENCES OF BOUNDED MEAN OSCILLATION

In this Chapter we study the space BMO(Z) consisting of sequences of bounded mean oscillation. This space plays an important role in Chapter V where we study the commutators. All the results in this Chapter are analogues of the corresponding results for the space BMO($\mathbb{R}$)([19], [32]). The proofs carry over to the discrete case of Z, by taking into account that the process of subdividing intervals terminates as soon as we have a singleton, a situation that does not occur in $\mathbb{R}$.We give the proofs only for the sake of completeness.

§ 1. The sequences of bounded mean oscillation.

4.2 Definition. For a sequence $a = \{a(n)\}$, we define the sharp maximal sequence $\mathbf{a}^{\#} = \{a^{\#}(n)\}$ as,

$$a^{\#}(n) = \sup_{n \in I} \frac{1}{\text{card } I} \sum_{k \in I} |a(k) - a_I|,$$

where the supremum is taken over all the intervals which contain n and $a_I = \frac{1}{\text{card } I} \sum_{k \in I} a(k)$.

A sequence $\mathbf{b} = \{b(n)\}$ is said to be in BMO(Z) if $\mathbf{b}^{\#} \in \ell^{\infty}$. If $\mathbf{b}$ is a constant sequence, then $b^{\#} = 0$. We define a norm on BMO(Z)/$\mathbb{C}$ as $\|\mathbf{b}\|_{*} = \|\mathbf{b}^{\#}\|_{\infty}$. Then this space is a Banach space.

4.3. Remarks. (i) If $\mathbf{b} = \{b(n)\}$ is a sequence and for each

interval I, there exists a constant C_I such that

$$\sup_I \frac{1}{\text{card } I} \sum_{k\in I} |b(k) - C_I| < \infty, \quad \text{then } \mathbf{b} \in BMO(Z).$$

(ii) ℓ^∞ is properly contained in BMO(Z).

An example of a sequence which is in BMO(Z), but not in ℓ^∞ is

$$\{b(n)\} = \begin{cases} \log|n| & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

Let $I = [m, m+1, \ldots, n]$ be any interval in Z. For the sake of illustration, assume $m, n > 0$. Now choosing $C_I = \log n$,

$$\begin{aligned}\sum_{k\in I} \big|\log |k| - \log n\big| &= \sum_{k=m}^{n} (\log n - \log k) \\ &\leq \int_m^n (\log n - \log t)\, dt + (\log n - \log m) \\ &= (n - m) - (m - 1)(\log n - \log m).\end{aligned}$$

Therefore

$$\frac{1}{n-m} \sum_{k=m}^{n} (\log n - \log k) \leq 1 - \left(\frac{m-1}{n-m}\right) [\log n - \log m] \leq 1.$$

4.3. <u>Calderón - Zygmund (C - Z) decomposition.</u>

Now we state and prove the discrete version of Calderón-Zygmund decomposition, which will be used to prove the John-Nirenberg inequality and the sharp maximal sequence theorem.

<u>Theorem.</u> Let $1 \leq p < \infty$ and $\mathbf{a} \in \ell^p$. For every $t > 0$, there exists a sequence of disjoint intervals $\{I_j^t\}$ such that

(i) $t < \dfrac{1}{\text{card } I_j^t} \displaystyle\sum_{k\in I_j^t} |a(k)| \leq 2t, \quad \forall\, j$

(ii) $\forall\, n \notin \bigcup_j I_j^t, \quad |a(n)| \leq t$

(iii) If $t_1 > t_2$, then each $I_j^{t_1}$ is subinterval of some $I_m^{t_2}$.

<u>**Proof.**</u> For each positive integer N, consider the collection of disjoint intervals of card 2^N,

$$\{I_{N,j}\}_{j\in Z} = \{[(j-1)2^N+1,\ldots,j2^N,\ j2^N-1]\}_{j\in Z}.$$

For each $t > 0$, let $N = N_t$ be the smallest positive integer such that

$$\frac{1}{\text{card } I_{N,j}} \sum_{k\in I_{N,j}} |a(k)| \leq t \qquad \forall\ j \text{ ———— } (1)$$

Now subdivide each of these intervals into two intervals of equal cardinality. If I is one of these intervals either

$$\text{(A)} \quad \frac{1}{\text{card } I} \sum_{k\in I} |a(k)| > t$$

$$\text{or} \quad \text{(B)} \quad \frac{1}{\text{card } I} \sum_{k\in I} |a(k)| \leq t.$$

In case (A) we select this interval and include it in a collection $\{I_j^t\}$. In case (B) we subdivide I once again unless I is a singleton, and select intervals as above.

Now the elements which are not included in $\{I_j^t\}$ form a set S such that for every $n \in S$ $|a(n)| \leq t$. This proves (ii).

Also from the choice of $\{I_j^t\}$, $\{I_j^t\}$ are disjoint and satisfy

$$\frac{1}{\text{card } I_j^t} \sum_{k\in I_j^t} |a(k)| > t.$$

Since each I_j^t is contained in an interval J_0 with card $J_0 = 2 \text{ card } I_j^t$, which is not selected in the previous step, we have

$$\frac{1}{\text{card } I_j^t} \sum_{k\in I_j^t} |a(k)| \leq \frac{2}{\text{card } J_0} \sum_{k\in J_0} |a(k)| \leq 2t.$$

This proves (i). It remains to prove (iii).

If $t_1 > t_2$ then $N_{t_1} \le N_{t_2}$. So each $I_{N_{t_1},j}$ is contained in some $I_{N_{t_2},m}$. In the subdividing and the selecting process for t_1, we have

$$\frac{1}{\operatorname{card} I_j^{t_1}} \sum_{k \in I_j^{t_1}} |a(k)| > t_1 > t_2,$$

so if $I_j^{t_1}$ is not one of the intervals $I_m^{t_2}$, then it must be a subinterval of some $I_m^{t_2}$ selected in an earlier step.

This completes the proof.

4.4. John Nirenberg's inequality.

One of the most important results about BMO is the John-Nirenberg inequality (see [32] p.202). As a consequence we get a family of equivalent norms on BMO(Z).

Theorem. Let $\mathbf{b} \in$ BMO(Z). Then there exist constants $C_1, C_2 > 0$ such that, for every finite interval I in Z and $\lambda > 0$,

$$\frac{\operatorname{card}\{ n \in I : |b(n) - b_I| > \lambda \}}{\operatorname{card} I} \le C_1 \, e^{-C_2 \lambda / \|\mathbf{b}\|_*} .$$

Proof. The key to the proof of the theorem is the C-Z decomposition restricted to an interval in Z.

First note that due to the homogeneity in the inequality we have to prove, one can assume that $\|\mathbf{b}\|_* = 1$. Fix an interval I in Z and apply C-Z decomposition (restricted to I) to the sequence $\{|b(n) - b_I|\}$ with $t = 3/2$. We get disjoint subintervals $\{I_j^{(1)}\}$ of I such that

(i) $|b_{I_j^{(1)}} - b_I| \le 3, \quad \forall j$

(ii) $|b(n) - b_I| \le 3/2, \quad \forall\, n \in I - \bigcup_j I_j^{(1)}$

(iii) $$\sum_j \text{card } I_j^{(1)} \le 2/3 \sum_j \sum_{k \in I_j^{(1)}} |b(k) - b_I|$$
$$\le 2/3 \sum_{k \in I} |b(k) - b_I|$$
$$\le 2/3 \text{ card } I.$$

Next we apply C-Z decomposition to each sequence $\{|b(n) - b_{I_j^{(1)}}|\}$ on $I_j^{(1)}$ with $t = 3/2$ unless this interval is just a singleton. We get disjoint intervals $\{I_j^{(2)}\}$ such that each $I_j^{(2)}$ is contained in some $I_m^{(1)}$. Also we have

(iv) $$|b_{I_j^{(2)}} - b_I| \le |b_{I_j^{(2)}} - b_{I_m^{(1)}}| + |b_{I_m^{(1)}} - b_I|$$
$$\le 6 \quad \forall\, j.$$

Note that those intervals, which are singletons in the collection $\{I_j^{(1)}\}$ belong to $I - \bigcup_j I_j^{(2)}$, since, for such an element n, $b(n) - b_{I_j^{(1)}} = 0$, and by (i) we have

(v) for every $n \in I - \bigcup_j I_j^{(2)}$,

$$|b(n) - b_I| \le 3/2 + 3 < 6.$$

Also we have :

(vi) $$\sum_j \text{card } I_j^{(2)} \le 2/3 \sum_j \text{card } I_j^{(1)}$$
$$\le (2/3)^2 \text{card } I.$$

Continuing this process, after some finite stage, say at the N-th stage, we will obtain intervals $\{I_j^{(N)}\}$, each of which is a singleton such that each $I_j^{(N)}$ is contained in some $I_m^{(N-1)}$ and the

singletons in the collection $\{I_m^{(N-1)}\}$ belong to $I - \bigcup_j I_j^{(N)}$. Also we have

(vii) For each $n \in I$,

$$|b(n) - b_I| < 3N$$

and (viii) $\sum_j \text{card } I_j^{(N)} \leq (2/3)^N \text{ card } I.$

Now if $3n \leq \lambda < 3n + 3$, $n = 1,2,\ldots,N-1$, then

$$\begin{aligned} \text{card } \{k \in I : |b(k) - b_I| > \lambda\} &\leq \sum_j \text{card } I_j^{(n)} \\ &\leq (2/3)^n \text{ card } I \\ &\leq e^{-C\lambda} \text{ card } I, \end{aligned}$$

for $C = (1/6) \log 3/2$.

Thus the theorem is proved if $3 \leq \lambda < 3N$. If $0 < \lambda < 3$, then

$$\begin{aligned} \text{card } \{k \in I : |b(k) - b_I| > \lambda\} &\leq \text{card } I \\ &\leq e^{3C} e^{-C\lambda} \text{ card } I. \end{aligned}$$

If $\lambda \geq 3N$, then $\text{card } \{k \in I : |b(k) - b_I| > \lambda\} = 0$.

Hence the proof of the theorem is complete.

4.5. John - Nirenberg theorem has an interesting corollary, namely, the reverse Hölder's inequality.

Corollary. Let $b \in BMO(Z)$. Then for every finite $p > 1$,

$$\sup_I \left(\frac{1}{\text{card } I} \sum_{k \in I} |b(k) - b_I|^p\right)^{1/p} \leq C_p \|b\|_*.$$

Proof. $\left(\frac{1}{\text{card } I} \sum_{k \in I} |b(k) - b_I|^p\right)$

$$= p \int_0^\infty \frac{\lambda^{p-1} \text{ card } \{ n \in I : |b(n) - b_I| > \lambda \}}{\text{card } I} \, d\lambda$$

$$\leq \quad C_1\ p \int_0^\infty \lambda^{p-1}\ e^{-C_2\lambda/\|b\|_*}\ d\lambda$$

$$= \quad C_1/C_2^p\ p\ \|b\|_*^p \int_0^\infty \lambda^{p-1}\ e^{-\lambda}\ d\lambda$$

$$= \quad C_1/C_2^p\ p\ \Gamma_p\ \|b\|_*^p$$

$$= \quad C_p\ \|b\|_*^p$$

This completes the proof of the corollary.

4.6. Some results on BMO(Z).

(1) Let $b \in BMO(Z)$ and I, J two finite intervals in Z, $I \subseteq J$.

(a) If $\operatorname{card} J \leq 2 \operatorname{card} I$, then $|b_I - b_J| \leq 2\|b\|_*$

(b) If $\operatorname{card} J > 2 \operatorname{card} I$, then

$$|b_I - b_J| \leq C \log [\frac{\operatorname{card} J}{\operatorname{card} I}]\ \|b\|_*.$$

Proof. (a) $|b_I - b_J| = |\frac{1}{\operatorname{card} I} \sum_{k\in I} [b(k) - b_J]|$

$$\leq \frac{2}{\operatorname{card} J} \sum_{k\in J} |b(k) - b_J|$$

$$\leq 2\|b\|_*.$$

(b) Let $I = I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n = J$, where $I_1, \ldots, I_n$ are intervals in Z such that $\operatorname{card} I_{k+1} \leq 2 \operatorname{card} I_k$ and where $n \leq C \log (\frac{\operatorname{card} J}{\operatorname{card} I})$. Then (a) gives

$$|b_I - b_J| \leq 2\ n\ \|b\|_*$$

$$\leq C \log (\frac{\operatorname{card} J}{\operatorname{card} I})\ \|b\|_*$$

(2) Let $b \in BMO(Z)$, I any interval in Z and n_0 the centre of I. Then for each $r > 1$, there exists a constant C_r such that

$$\left(\sum_{n \notin 2I} \frac{|b(n) - b_I|^r}{|n_0 - n|^r} \right)^{1/r} \leq \frac{C_r \|b\|_*}{(\text{card } I)^{1/r'}}$$

where $1/r + 1/r' = 1$.

Proof. For $k = 0,1,2,\ldots,$ let $I_k = 2^k I$ and $J_k = I_{k+1} \setminus I_k$

Now

$$\left(\sum_{n \notin 2I} \frac{|b(n) - b_I|}{|n_0 - n|^r} \right)^{1/r} = \left(\sum_{k=1}^{\infty} \sum_{n \in J_k} \frac{|b(n) - b_I|^r}{|n_0 - n|^r} \right)^{1/r}$$

$$\leq \left(\sum_{k=1}^{\infty} \sum_{n \in J_k} \frac{|b(n) - b_I|^r}{2^{kr} (\text{card } I)^r} \right)^{1/r}$$

$$\leq \left(\sum_{k=1}^{\infty} \sum_{n \in J_k} \frac{|b(n) - b_{I_{k+1}}|^r}{2^{kr} (\text{card } I)^r} \right)^{1/r}$$

$$+ \left(\sum_{k=1}^{\infty} \sum_{n \in J_k} \frac{|b_I - b_{I_{k+1}}|^r}{2^{kr} (\text{card } I)^r} \right)^{1/r}$$

$$= A_1 + A_2 \text{ say.}$$

Now by Cor. 4.5 we have

$$A_1 \leq \left(\sum_{k=1}^{\infty} \frac{4}{2^{(r-1)k} (\text{card } I)^{r-1}} \frac{1}{\text{card } I_{k+1}} \sum_{n \in I_{k+1}} |b(n) - b_{I_{k+1}}|^r \right)^{1/r}$$

$$A_1 \leq \left(\sum_{k=1}^{\infty} \frac{4}{2^{(r-1)k} (\text{card } I)^{r-1}} \| b \|_*^r \right)^{1/r}$$

$$\leq \frac{C}{(\text{card } I)^{1/r'}} \| b \|_*$$

$$\text{And} \quad A_2 \le \Big(\sum_{k=1}^{\infty} \sum_{n \in I_{k+1} - I_k} \frac{[\log(\frac{\text{card } I_{k+1}}{\text{card } I})]^r \, \|b\|_*^r}{2^{rk} \quad (\text{card } I)^r} \Big)^{1/r}$$

$$\le \Big(\sum_{k=1}^{\infty} \frac{2^{k+1}(\text{card } I)(\log 2^{k+3})^r \, \|b\|_*^r}{2^{rk} \, (\text{card } I)^r} \Big)^{1/r}$$

$$\le \frac{\|b\|_* \quad 2^{1/r}}{(\text{card } I)^{1/r'}} \Big(\sum_{k=1}^{\infty} \frac{(k+3)^r}{2^{(r-1)k}} \Big)^{1/r}$$

$$\le \frac{C \, \|b\|_*}{(\text{card } I)^{1/r'}} .$$

4.7. Next we state and prove the sharp maximal sequence theorem which will be used in studying commutators. The proof is the discretized version of the corresponding result on $\mathbb{R}$ and is given only for the sake of completeness.

Theorem. Let $1 < p < \infty$. Then there exists a constant $C_p > 0$ such that $\| M\, \mathbf{a} \|_p \le C_p \| \mathbf{a}^{\#} \|_p$, $\forall\, \mathbf{a} \in \ell^p$.

Proof. We apply the C - Z decomposition to the sequence a. For every $t > 0$, we get a sequence of disjoint intervals $\{J_k^t\}$ satisfying (i), (ii) and (iii) of Theorem 4.3. We prove the theorem in three steps.

Step 1. Define $\alpha(t) = \sum_k \text{card } J_k^t$. We will prove that for any $A>0$,

$$\alpha(t) \le \text{card } \{n : \mathbf{a}^{\#}(n) > t/A\} + 2/A \; \alpha(t/4).$$

Fix an interval $J_m^{t/4} = I_0$ and $A > 0$. Suppose

$$I_0 \subseteq \{n : \mathbf{a}^{\#}(n) > t/A\},$$

then, we have

$$\sum_{\{j : J_j^t \subseteq I_0\}} \text{card } J_j^t \le \text{card } [\{n : \mathbf{a}^{\#}(n) > t/A\} \cap I_0].$$

If I_0 is not contained in $\{n : \mathbf{a}^{\#}(n) > t/A\}$, then, there exists $j \in I_0$ such that $\mathbf{a}^{\#}(j) \le t/A$. Since $j \in I_0$,

$$\frac{1}{\text{card } I_0} \sum_{k \in I_0} |a(k) - a_{I_0}| \le t/A.$$

Also observe that $|\mathbf{a}|_{I_0} \le 2\ t/4 = t/2$, where $|\mathbf{a}|(k) = |a(k)|$.

Therefore

$$\begin{aligned}
\frac{t}{2} \sum_{j : J_j^t \subseteq I_0\}} \text{card } J_j^t &= \sum_{\{j : J_j^t \subseteq I_0\}} [t \text{ card } J_j^t - t/2 \text{ card } J_j^t] \\
&\le \sum_{\{j : J_j^t \subseteq I_0\}} [\sum_{k \in J_j^t} |a(k)| - |a|_{I_0}] \\
&\le \sum_{\{j : J_j^t \subseteq I_0\}} [\sum_{k \in J_j^t} |a(k) - a_{I_0}|] \\
&\le \sum_{k \in I_0} |a(k) - a_{I_0}| \\
&\le \frac{t}{A} \text{ card } I_0 .
\end{aligned}$$

Since each J_j^t is contained in some $J_m^{t/4}$, summing over all J_j^t' s which are contained in I_0 , we get

$$\sum_j \text{card } J_j^t \le \text{card } \{n : a^{\#}(n) > t/A\} + 2/A\ \alpha(t/4).$$

<u>Step 2</u>. Let $\beta(t) = \text{card } \{n : M\ a(n) > t\}$.

First our claim is $\alpha(t) \le \beta(t)$. For each j, we have

$$\frac{1}{\text{card } J_j^t} \sum_{k \in J_j^t} |a(k)| > t.$$

Therefore $\bigcup_j J_j^t \subseteq \{n : M\,a(n) > t\}$, which implies $\alpha(t) \le \beta(t)$.

Our second claim is that there exist constants $C_1, C_2 > 0$ such that $\beta(t) \le C_1\ \alpha(C_2 t)$. Let $n \notin \bigcup_j 2\ J_j^t$ and I be any interval which contains n. Then

$$\sum_{k\in I} |a(k)| = \sum_{k\in I\cap(\bigcup_j J_j^t)} |a(k)| + \sum_{k\in I\cap(\bigcup_j J_j^t)^C} |a(k)|$$

$$= S_1 + S_2 \text{ say.}$$

Since $|a(k)| \le t$ for $k \in (\bigcup_j J_j^t)^C$, $S_2 \le t$ card I.

To estimate S_1, we observe a simple geometric fact. If $I \cap J_j^t$ is non-empty and I is not contained in $2\ J_j^t$, then $J_j^t \subseteq 4I$. Since $n \in I$ and $n \notin 2J_j^t$ for each j, I is not contained in $2J_j^t$ for each j.

Therefore
$$S_1 \le \sum_{\{j:J_j^t \subseteq 4I\}} \sum_{k\in J_j^t} |a(k)|$$

$$\le \sum_{\{j:J_j^t \subseteq 4I\}} 2\ t \text{ card } J_j^t$$

$$\le 2\ t \text{ card } 4\ I.$$

$$\le 8\ t \text{ card } I.$$

Therefore $\sum_{k\in I} |a(k)| \le S_1 + S_2 \le 9\ t$ card I.

Since I was an arbitrary interval containing n, we have $Ma(n) \le 9t$.

Therefore $(\bigcup_j 2\ J_j^t)^C \subseteq \{n : M\,a(n) \le 9\ t\}$

which implies $\beta(9t) \le 2\ \alpha(t)$ for each $t > 0$, hence $\beta(t) \le 2\ \alpha(t/9)$ for each $t > o$ and this proves our claim.

Step 3. Define $I_N = p \int_0^N t^{p-1} \alpha(t)\, dt.$

Since $\mathbf{a} \in \ell^p$, I_N is finite for each N. We claim that $I_N \le C_p \| \mathbf{a}^\# \|_p^p$.

$$I_N = p \int_0^N t^{p-1} \alpha(t)\, dt$$

$$\le p \int_0^N t^{p-1} \text{card}\{n : \mathbf{a}^\#(n) > t/A\} dt + 2/A\ p \int_0^N t^{p-1} \alpha(t/4)\, dt$$

Now $$p \int_0^N t^{p-1} \text{card}\{n : \mathbf{a}^\#(n) > t/A\}\, dt$$

$$\le p \int_0^\infty t^{p-1} \text{card}\{n : \mathbf{a}^\#(n) > t/A\}\, dt$$

$$\le A^p\ p \int_0^\infty t^{p-1} \text{card}\{n : \mathbf{a}^\#(n) > t\}\, dt$$

$$= A^p \|\mathbf{a}^\#\|_p^p.$$

Also $$p \int_0^N t^{p-1} \alpha(t/4)\, dt = 4^p\ p \int_0^N t^{p-1} \alpha(t)\, dt \le 4^p\ I_N.$$

Therefore $I_N \le A^p \|\mathbf{a}^\#\|_p^p + 2/A\ 4^p\ I_N$.

Now choosing $A = 4^{p+1}$, we get $I_N \le C_p \|\mathbf{a}^\#\|_p^p$, which proves our claim.

Therefore $$\sum_{n \in Z} [M\, a(n)]^p = p \int_0^\infty t^{p-1} \beta(t)\, dt \le C_1\ p \int_0^\infty t^{p-1} \alpha(C_2 t)\, dt$$

$$\leq C_1/C_2^p \; p \int_0^\infty t^{p-1} \alpha(t)\, dt$$

$$= C \lim_{N \to \infty} I_N$$

$$\leq C_p \|a^{\#}\|_p^p .$$

Hence the proof of the theorem is complete.

4.8. Remark. We would like to emphasize that the fact that $\mathbf{a} \in \ell^p$ is needed in the proof (step 3) of theorem 4.7. Therefore, if for a sequence $\mathbf{a}$ we have $a^{\#} \in \ell^p$, and $\mathbf{a} \in \ell^p$, then the inequality $\|\mathbf{a}\|_p \leq C \|\mathbf{a}^{\#}\|_p$ holds.

§ 2. THE SPACE $h^1_{at}(Z)$ AND ITS DUAL.

In this section we define the atomic Hardy space h^1_{at} on Z and show that it is the predual of BMO(Z).

4.9. Definition. A real valued sequence $\mathbf{a} = \{a(n)\}$ is said to be a $(1,\infty)$ atom on Z if

(i) Support of $\mathbf{a}$ is contained in some finite interval I.

(ii) $\sum_{n \in I} a(n) = 0$

(iii) $\|\mathbf{a}\|_\infty \leq \dfrac{1}{\text{card } I}$

We define $h^1_{at} = \{f = \sum c_i \mathbf{a}_i : \mathbf{a}_i\text{'s are } (1,\infty) \text{ atoms}, \sum |c_i| < \infty\}$ with norm

$$\|f\|_{h^1_{at}} = \inf \{\textstyle\sum |c_i| : f = \sum c_i \mathbf{a}_i,\ \mathbf{a}_i\text{'s are } (1,\infty) \text{ atoms}, \sum |c_i| < \infty\}.$$

Then h^1_{at} is a Banach space.

4.10. Proposition. Let $\mathbf{b} \in BMO(Z)$. Then b defines a continuous linear functional on $h^1_{at}(Z)$ with norm less than or equal to $\|b\|_*$.

Proof. Let $\mathbf{a}$ be a $(1,\infty)$ atom supported in an interval I. Then

$$\Big|\sum_{k\in Z} b(k)a(k)\Big| = \Big|\sum_{k\in Z} [b(k) - b_I]\, a(k)\Big|$$

$$\le \|\mathbf{a}\|_\infty \sum_{k\in I} |b(k) - b_I|$$

$$\le \frac{1}{\text{card } I} \sum_{k\in I} |b(k) - b_I| \le \|\mathbf{b}\|_*.$$

Therefore if $f = \sum_{i=1}^{N} \lambda_i \mathbf{a}_i$, where $\mathbf{a}_i$'s are $(1,\infty)$ atoms, then,

$$|\langle \mathbf{b}, f \rangle| = \Big|\sum_{k\in Z} b(k)f(k)\Big|$$

$$\le \|\mathbf{b}\|_* \sum_{i=1}^{N} |\lambda_i|.$$

Since the set $\{f = \sum_{i=1}^{N} \lambda_i a_i : a_i\text{'s are } (1,\infty) \text{ atoms and } N \in Z_+\}$ is dense in $h^1_{at}(Z)$, we conclude the proposition.

4.11. Now we prove the converse.

Theorem. Let L be a continuous linear functional on $h^1_{at}(Z)$. Then there exists a sequence $\mathbf{b} = \{b(n)\} \in BMO(Z)$ such that if $f = \sum_{i=1}^{N} \lambda_i \mathbf{a}_i$, where $\mathbf{a}_i$s are $(1,\infty)$ atoms, then

$$\Big|\sum_{k\in Z} b(k)a(k)\Big| \le \|L\|\, \|f\|_{h^1_{at}}.$$

Proof. Let I be a finite interval in Z and

$$\ell^\infty_0(I) = \{g \in \ell^\infty(I) : \sum_{k\in I} g(k) = 0\}.$$

If $g \in \ell_0^\infty(I)$, then $\dfrac{g}{(\text{card } I)\|g\|_\infty}$ is a $(1,\infty)$ atom, so that

$$\|g\|_{h^1_{at}} \leq (\text{card } I)\ \|g\|_\infty \text{ and } |L(g)| \leq \|L\|\ (\text{card } I)\ \|g\|_\infty .$$

Hence L defines a continuous linear functional on $\ell_0^\infty(I)$. By the Hahn - Banach theorem, L can be extended to a continuous linear functional L_1 on $\ell^\infty(I)$ with the same norm. Since I is finite, there exists $b^{(I)} \in \ell^1(I)$ such that $L_1 g = \sum_{k\in I} b^{(I)}(k)\ g(k)$, for each $g \in \ell^\infty(I)$. We now, estimate $\| b^{(I)} - b_I^{(I)}\|_{\ell^1(I)}$.

Let $g \in \ell^\infty(I)$ such that $\|g\|_\infty \leq 1$. Since

$$\sum_{k\in I} [b^{(I)}(k) - b_I^{(I)}]\ g_I = \sum_{k\in I} [g(k) - g_I]\ b_I^{(I)} = 0,$$

We have

$$\begin{aligned}
\Big|\sum_{k\in I} [b^{(I)}(k) - b_I^{(I)}]\ g(k)\Big| &= \Big|\sum_{k\in I} [b^{(I)}(k) - b_I^{(I)}]\ [g(k) - g_I]\Big| \\
&= \Big|\sum_{k\in I} [g(k) - g_I]\ b^{(I)}(k)\Big| \\
&= |L_1(g - g_I)| \\
&= |L(g - g_I)| \\
&\leq 2\ \|L\|\ (\text{card } I).
\end{aligned}$$

This implies $\|b^{(I)} - b_I^{(I)}\|_{\ell^1(I)} \leq 2\ \|L\|\ (\text{card } I)$. Therefore

$$\frac{1}{\text{card } I} \sum_{k\in I} |b^{(I)}(k) - b_I^{(I)}| \leq 2\ \|L\|.$$

Now let I_1 and I_2 be two finite intervals such that $I_1 \subseteq I_2$. Let $b^{(I_1)} \in \ell^1(I_1)$ and $b^{(I_2)} \in \ell^1(I_2)$ be the sequences found above for these intervals, then

$$b^{(I_2)}|_{I_1} - b^{(I_1)} = \text{constant on } I_1 .$$

Let $n, m \in I_1$ such that $n \neq m$. Consider the sequence

$$g(k) = \begin{cases} 1 & \text{if } k = m \\ -1 & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Since the sequence $g \in \ell_0^\infty(I_1) \cap \ell_0^\infty(I_2)$,

$L(g) = \sum_{k \in I} b^{(I_1)}(k)\, g(k) = \sum_{k \in I_2} b^{(I_2)}(k)\, g(k)$, which implies

$b^{(I_1)}(m) - b^{(I_2)}(m) = b^{(I_1)}(n) - b^{(I_2)}(n)$.

Since m and n are arbitrary elements in I_1, we conclude that

$b^{(I_2)}|_{I_1} - b^{(I_1)} =$ constant on I_1.

For $k = 1,2,3,\ldots$ let $I_k = [-2^k, \ldots, 2^k]$ and $b^{(I_k)} \in \ell^1(I_k)$ be the corresponding sequences found above for these intervals.By the above argument, for each $k \in Z$, $b^{(I_{k+1})}|_{I_k} - b^{(I_k)} = C_k$, a constant on I_k. Define the sequence $\mathbf{b} = \{b(j)\}$ by

$$\begin{aligned} b(j) &= b^{(I_1)}(j) & \text{if } j \in I \\ &= b^{(I_{k+1})}(j) - (C_1 + C_2 + \ldots + C_k) & \\ & & \text{if } j \in I_{k+1} \text{ for } k = 1,2,\ldots \end{aligned}$$

It remains to prove that $b \in BMO(Z)$. Let I be any interval in Z and $b^{(I)} \in \ell^1(I)$ be the corresponding sequence for I. Since

$$\frac{1}{\text{card } I} \sum_{k \in I} |b^{(I)}(k) - b_I^{(I)}| \leq 2\, \|L\| \text{ and } \mathbf{b} - b^{(I)} \text{ is}$$

constant on I, $\frac{1}{\text{card } I} \sum_{k \in I} |b(k) - b_I| \leq 2\, \|L\|$. This proves that $b \in BMO(Z)$ and $\|\mathbf{b}\|_* \leq 2\|L\|$.

Also if a is a $(1,\infty)$ atom and support of a is contained in I_k for

sufficiently large k, we have

$$\left|\sum_{j\in Z} b(j)\ a(j)\right| = \left|\sum_{j\in Z} b^{(I_k)}(j)\ a(j)\right| \le \| L \| .$$

Hence if $f = \sum_{i=1}^{N} \lambda_i \mathbf{a}_i$, where $\mathbf{a}_i$'s are $(1,\infty)$ atoms then

$$\left|\sum_{j\in Z} b(j)\ f(j)\right| \le \|L\| \sum_{i=1}^{N} |\lambda_i| .$$

Hence the proof of the theorem is complete.

4.12. The h^1 spaces have a maximal function characterization, which we now describe. This is implicitly contained in [9], where the authors have given the atomic decomposition of the ergodic Hardy space $H^1(X)$.

Definition. Let ψ be a differentiable function on $(-1,1)$ with $0 \le \psi \le 1$ and $\| \psi '\|_\infty < \infty$. For a sequence $a = \{a(n)\}$, we define

$$a_\psi^*(k) = \sup_{|i|<n} |a * \psi_n (i + k)| , \text{ where } \psi_n(m) = 1/n\ \psi(m/n).$$

and define $a^*(k) = \sup_\psi [a_\psi^*(k)\ A(\psi)^{-1}]$, where

$A(\psi) = \| \psi \|_\infty + \| \psi '\|_\infty$.

Since $\psi \le \chi_{[-1,1]}$, it is easy to see that $\mathbf{a}^* \le M\ \mathbf{a}$, hence $\mathbf{a}^*$ is bounded on ℓ^p for $1 < p < \infty$ and satisfies a weak $(1,1)$ inequality.

4.13. **Definition.** We define $h^1_{max} = \{\mathbf{a} \in \ell^1 : \mathbf{a}^* \in \ell^1\}$ with the norm $\| \mathbf{a} \|_{h^1_{max}} = \| \mathbf{a}^* \|_1$.

In the following theorem the maximal function characterization of h^1_{at} spaces is given.

4.14. Theorem[9]. (i) Let $1 < q \le \infty$. If $a \in h^1_{at}$, then $a \in h^1_{max}$ and $\| a \|_{h^1_{max}} \le C(p,q) \| a \|_{h^1_{at}}$.

(ii) Conversely, if $f \in h^1_{max}$, then, there exists a sequence $\{a_i\}$ of $(1,\infty)$ atoms such that $a = \sum_i C_i a_i$ and

$$\sum_i |C_i| < C \|a\|_{h^1_{max}} .$$

4.15. Definition. We define $h^1_{con} = \{a \in \ell^1 : Ha \in \ell^1\}$ with norm $\|a\|_{h^1_{con}} = \|a\|_1 + \|Ha\|_1$.

Now we prove that $h^1_{at} \subseteq h^1_{con}$. The proof is similar to the proof of part (i) of Theorem 4.13 [9]. For the sake of completeness we give the proof.

4.16. Proposition. If $a \in h^1_{at}$, then $a \in h^1_{con}$ and

$$\| Ha \|_1 \le C \| a \|_{h^1}.$$

Proof. In order to prove the Proposition, it is enough to prove $\| Ha \|_1 \le C$, for every $(1,\infty)$ atom a. Since H commutes with translations, we can assume that supp $a \subseteq [0,1,\dots,K-1]$.

$$\text{Now } \sum_{m=-\infty}^{\infty} |H a(m)| = \sum_{m=-4K}^{4K} |Ha(m)| + \sum_{|m|>4K} |Ha(m)|$$

$$= S_1 + S_2 \text{ say.}$$

$$\text{Now } S_1 \le (8K+1)\,[1/(8K+1) \sum_{m=-4K}^{4K} |H a(m)|^q]^{1/q} \text{ for some } q>1$$

$$\le C\,(8K+1)^{1-1/q}\,[\sum_{m=-\infty}^{\infty} |a(m)|^q]^{1/q}$$

$$\le C\,(8K+1)^{1-1/q}\,K^{-(1-1/q)} \le C.$$

Now for $|m| > 4K$,

$$|H\,\mathbf{a}(m)| = \left|\sum_{j=0}^{K-1} a(j)/(m-j)\right|$$

$$= \left|\sum_{j=0}^{K-1} a(j)\,[1/m - 1/(m-j)]\right|$$

$$\le C \sum_{j=0}^{K-1} (|a(j)|)\,|j|/m^2$$

$$\le C\ K/m^2 \sum_{j=0}^{K-1} |a(j)|$$

$$\le C\ K/m^2$$

Therefore $S_2 \le C\,K \sum_{|m|>4K} 1/m^2$

$$\le C.$$

Hence the proof of the Proposition is complete.

4.17. **Remark.** We have not been able to prove that $h^1_{at} = h^1_{con}$. The corresponding result for $\mathbb{R}$ or $\mathbb{T}$ is well-known (see[32]). For ergodic Hardy spaces, we refer to [9] and [13].

CHAPTER V

COMMUTATORS

In this Chapter we study the commutator of the operator of pointwise multiplication by a sequence $\mathbf{b} = \{b(n)\}$ and an S-operator. More precisely, we consider the operator given by

$$([\mathbf{b}, T_\phi]\mathbf{a})(n) = b(n)\, T_\phi \mathbf{a}(n) - T_\phi(\mathbf{b}\mathbf{a})(n)$$

$$= \sum_{k=-\infty}^{\infty} \phi(k)\,[b(n) - b(n-k)]\, a(n-k).$$

Let $1 < p < \infty$. We ask the question: For which sequences $\mathbf{b}=\{b(n)\}$, is the commutator bounded on ℓ^p? This question is answered in section 1. In section 2 we study the commutator operator of ergodic S-operators on the spaces $L^p(X)$ where $(X,\mathbb{B},m)$ is a probability space equipped with an invertible measure preserving transformation U. For the commutator on the spaces $L^p(\mathbb{R})$, $1 < p < \infty$, we refer to [11].

§ 1. COMMUTATORS ON SEQUENCE SPACES

We begin this section by proving that if T_ϕ is the discrete Hilbert transform H, we prove that the boundedness of $[\mathbf{b}, H]$ implies that $\mathbf{b} \in BMO(Z)$. A similar result holds for $\mathbb{R}$ (see [11]).

5.1. Theorem. Let $1 \le p < \infty$. Suppose there exists a constant $C_p > 0$ such that $\|[\mathbf{b},H]\mathbf{a}\|_p \le C_p \|\mathbf{a}\|_p$, $\forall\, \mathbf{a} \in \ell^p$. Then $\mathbf{b} \in BMO(Z)$.

Therefore

$$\operatorname*{Sup}_{I} \frac{1}{\text{Card } I} \sum_{n \in I} |b(n) - b_I| \le C.$$

This completes the proof.

5.2 The converse of Theorem 5.1 is also true, namely, if $\mathbf{b} \in BMO(Z)$, then the commutator of the Hilbert transform is bounded on ℓ^p for $1 < p < \infty$. In fact this is true for any S-operator.

In section 2, we will study the commutator problem for ergodic S-operators. For proving the existence of such operators and the existence of commutators of S-operators we need inequalities for the associated maximal operator.

Define,

$$[\mathbf{b}, T_\phi]^* \mathbf{a}(n) = \operatorname*{Sup}_{N} \left| \sum_{k=-N}^{N} \phi(k) \, [b(n) - b(n-k)] \, a(n-k) \right|.$$

Next we state and prove the maximal commutator theorem.

5.3 **Theorem.** Let $1 < p < \infty$ and $\mathbf{b} \in BMO(Z)$. Then there exists a constant $C_p > 0$ such that

$$\| [\mathbf{b}, T_\phi]^* \mathbf{a} \|_p \le C_p \, \|\mathbf{a}\|_p, \ \forall \ \mathbf{a} \in \ell^p.$$

We will prove Theorem 5.3 through several lemmas. The proof is by an application of the sharp maximal sequence theorem (see [4.7]).

We recall that an S-kernel satisfies,

S3. $|\phi(n-j) - \phi(n)| \leq \frac{C\,|j|}{(n-j)^2}$ for $|n| > 2\,|j|$.

As remarked earlier, the truncations ϕ_N of ϕ do not satisfy S3 uniformly in N. This fact plays an important role in the proof of Theorem 5.3. We will dominate the maximal commutator $[\mathbf{b},\, T_\phi]^*$ by the sum of two maximal operators $[\mathbf{b},\, T_\nu]^*$ and $[\mathbf{b},\, T_{|\psi|}]^*$. The sequences $\{\nu_N\}$ and $\{\psi_N\}$ which define these operators satisfy S3 uniformly. We define these below.

5.4 Definition. Consider the differentiable functions ν and ψ defined on $(0,\, \infty)$ by:

$$\nu(t) = \begin{cases} 1, \text{ if } 0 < t \leq 1/2 \\ \frac{1}{2}\,[1-\cos 2\pi t], \text{ if } \frac{1}{2} < t \leq 1 \\ 0, \text{ if } t > 1 \end{cases}$$

and

$$\psi(t) = \begin{cases} 0, \text{ if } 0 < t \leq 1/2 \\ \frac{1}{2}\,[1+\cos 2\pi t], \text{ if } \frac{1}{2} < t \leq 1 \\ 1, \text{ if } 1 < t < 2 \\ \frac{1}{2}\,[1-\cos \frac{\pi t}{2}\,], \text{ if } 2 \leq t \leq 4 \\ 0, \text{ if } t > 4 \end{cases}$$

We observe that

$$|\, \chi_{[0,1)}\,(t) - \nu(t)| \leq \psi(t),\; t \in (0,\, \infty). \tag{5.5}$$

For $j \in Z$, let

$$\nu_N(j) = \phi(j)\;\nu\;(\frac{|j|}{N})$$

$$\psi_N(j) = \phi(j)\;\psi\;(\frac{|j|}{N})\;.$$

Using the kernels $\{\nu_N\}$ and $\{\psi_N\}$ we define the operators $[\mathbf{b}\ T_\nu]^*$ and $[\mathbf{b},\ T_{|\psi|}]^*$ as

$$[\mathbf{b},\ T_\nu]^*\ \mathbf{a}(n) = \operatorname*{Sup}_{N\geq 1} \left| \sum_{j=-\infty}^{\infty} [b(n) - b(j)]\ \nu_N(n-j)\ a(j)\right|$$

and

$$[\mathbf{b},\ T_{|\psi|}]^*\ \mathbf{a}(n) = \operatorname*{Sup}_{N\geq 1} \sum_{j=-\infty}^{\infty} |[b(n) - b(j)]|\ |\psi_N(n-j)||a(j)|\ .$$

We remark that $[\mathbf{b},\ T_{|\psi|}]^*$ is not exactly a maximal commutator because the modulus occurs inside the summation.

In the following lemma, we show the relationship between $[\mathbf{b},\ T_\phi]^*$, $[\mathbf{b},\ T_\nu]^*$ and $[\mathbf{b},\ T_{|\psi|}]^*$.

5.6 Lemma. For each $n \in Z$,

$$[\mathbf{b},T_\phi]^*\ \mathbf{a}(n) \leq [\mathbf{b},T_\nu]^*\ \mathbf{a}(n) + [\mathbf{b},T_{|\psi|}]^*\ \mathbf{a}(n).$$

Proof. First note that, for $n \in Z$,

$$[\mathbf{b},T_\phi]^*\ \mathbf{a}(n) = \operatorname*{Sup}_{N\geq 1} \left| \sum_{j=-\infty}^{\infty} [b(n) - b(j)]\ \phi(n-j)\ \chi_{(0,1]}\left(\frac{|n-j|}{N}\right)\ a(j)\right|.$$

Therefore

$$\left|[\mathbf{b},T_\phi]^*\ \mathbf{a}(n) - [\mathbf{b},T_\nu]^*\ \mathbf{a}(n)\right|$$

$$= \left| \operatorname*{Sup}_{N\geq 1} \left| \sum_{j=-\infty}^{\infty} [b(n) - b(j)]\ \phi(n-j)\ \chi_{(0,1]}\left(\frac{|n-j|}{N}\right)\ a(j)\right| - \operatorname*{Sup}_{N\geq 1} \left| \sum_{j=-\infty}^{\infty} [b(n) - b(j)]\ \phi(n-j)\ \nu\left(\frac{|n-j|}{N}\right)\ a(j)\right|\right|$$

$$\leq \operatorname*{Sup}_{N\geq 1} \left| \sum_{j=-\infty}^{\infty} [b(n) - b(j)]\ \phi(n-j)\left[\chi_{(0,1]}\left(\frac{|n-j|}{N}\right) - \nu\left(\frac{|n-j|}{N}\right)\right]\ a(j)\right|$$

$$\leq \operatorname*{Sup}_{N\geq 1} \sum_{j=-\infty}^{\infty} |b(n) - b(j)|\ |\phi(n-j)|\ \left|\chi_{(0,1]}\left(\frac{|n-j|}{N}\right)-\nu\left(\frac{|n-j|}{N}\right)\right|\ |a(j)|.$$

Now using inequality (5.5), we get

$$|[\mathbf{b},T_\phi]^* \mathbf{a}(n) - [\mathbf{b},T_\nu]^* \mathbf{a}(n)|$$

$$\leq \sup_{N\geq 1} \sum_{j=-\infty}^{\infty} |b(n) - b(j)|\ |\phi(n-j)|\ |\psi(\tfrac{|n-j|}{N})|\ |a(j)|$$

$$= [\mathbf{b},T_{|\psi|}]^* \mathbf{a}(n).$$

Therefore, for each $n \in Z$,

$$[\mathbf{b},T_\phi]^* \mathbf{a}(n) \leq [\mathbf{b},T_\nu]^* \mathbf{a}(n) + [\mathbf{b},T_{|\psi|}]^* \mathbf{a}(n).$$

Hence the proof of Lemma 5.6 is complete.

Therefore, for proving Theorem 5.3, it is enough to prove that the operators $[\mathbf{b},T_\nu]^*$ and $[\mathbf{b},T_{|\psi|}]^*$ are bounded on ℓ^p. Towards this, we first prove that the kernels $\{\nu_N\}$ and $\{\psi_N\}$ satisfy condition S3 uniformly.

5.7 <u>**Lemma.**</u> There exists a constant $C > 0$ such that

$$|\nu_N(n-j) - \nu_N(n)| \leq \frac{C|j|}{(n-j)^2} \text{ for } |n| > 2|j| \text{ and } \forall\ N \geq 1$$

and

$$|\psi_N(n-j) - \psi_N(n)| \leq \frac{C|j|}{(n-j)^2} \text{ for } |n| > 2|j| \text{ and } \forall\ N \geq 1.$$

<u>**Proof.**</u> First consider the kernels $\{\nu_N\}$. Let $|n| > 2|j|$.

$$|\nu_N(n-j) - \nu_N(n)| = |\phi(n-j)\ \nu(\tfrac{|n-j|}{N}) - \phi(n)\ \nu(\tfrac{|n|}{N})|$$

$$\leq |\phi(n-j) - \phi(n)|\ |\nu(\tfrac{|n|}{N})|$$

$$+ |\phi(n-j)|\ |\nu(\tfrac{|n|}{N}) - \nu(\tfrac{|n-j|}{N})|$$

$$\leq \frac{C\,|j|}{(n-j)^2} + \frac{C}{|n-j|}\ |\ \nu(\tfrac{|n|}{N} - \nu(\tfrac{|n-j|}{N})|.$$

Since supp $\nu \subseteq (0,1]$ and $|n| > 2|j|$,

$$\left|\nu\left(\frac{|n-j|}{N}\right) - \nu\left(\frac{|n|}{N}\right)\right| = 0 \text{ if } \frac{|n-j|}{N} > 2.$$

If $\frac{|n-j|}{N} \leq 2$, applying the mean value theorem we get

$$\left|\nu\left(\frac{|n-j|}{N}\right) - \nu\left(\frac{|n|}{N}\right)\right| \leq \frac{|j|}{N}\, \nu'(t_o),$$

where t_o is a point between $\frac{|n-j|}{N}$ and $\frac{|n|}{N}$.

But $|\nu'(t)| \leq \pi$, $\forall\ t \in (0,\infty)$.

Therefore,

$$\left|\nu\left(\frac{|n-j|}{N}\right) - \nu\left(\frac{|n|}{N}\right)\right| \leq \frac{|j|}{N}\, \pi \ \leq \frac{2\pi|j|}{|n-j|} .$$

Hence the kernels $\{\nu_N\}$ satisfy condition S3 uniformly. Similarly the kernels $\{\psi_N\}$ satisfy condition S3 uniformly, since $|\psi'(t)| \leq \pi$, $\forall\ t \in (0,\infty)$ and supp $\psi \subseteq (0,4]$.

Hence the proof of Lemma 5.7 is complete.

5.8. For proving the boundedness of the operators $[\mathbf{b}, T_\nu]^*$ and $[\mathbf{b}, T_{|\psi|}]^*$ on ℓ^p, we need to consider the maximal operators T^*_ν and $T^*_{|\psi|}$, defined as :

$$T^*_\nu\ \mathbf{a}(n) = \sup_{N\geq 1} \left|\sum_{k=-\infty}^{\infty} \nu_N(n-k)\ a(k)\right|$$

and
$$T^*_{|\psi|}\ \mathbf{a}(n) = \sup_{N\geq 1} \sum_{k=-\infty}^{\infty} |\psi_N(n-k)|\ |a(k)|.$$

5.9. <u>**Lemma.**</u> Let $1 < p < \infty$. Then there exists a constant $C_p > 0$ such that

$$\|T^*_\nu\ \mathbf{a}\|_p \leq C_p\ \|\mathbf{a}\|_p,\ \forall\ \mathbf{a} \in \ell^p \tag{5.10}$$

and
$$\|T^*_{|\psi|}\ \mathbf{a}\|_p \leq C_p\ \|\mathbf{a}\|_p,\ \forall\ \mathbf{a} \in \ell^p \tag{5.11}$$

Proof. First let us prove 5.10.

For a non negative real number α, let $[\alpha]$ denote the greatest integer less than or equal to α. Then

$$\Big|\sum_{|n-j|\le N} \phi(n-j)\,\nu\left(\frac{|n-j|}{N}\right) a(j)\Big|$$

$$\le \Big|\sum_{|n-j|\le[N/2]} \phi(n-j)a(j)\Big| + \sum_{N\ge|n-j|>[N/2]} |\phi(n-j)|\ |a(j)|$$

$$\le \Big|\sum_{|n-j|\le[N/2]} \phi(n-j)a(j)\Big| + C_2 \sum_{N\ge|n-j|>[N/2]} \frac{|a(j)|}{|n-j|}$$

$$\le C\,[T_{\phi}^{*} a(n) + Ma(n)],$$

where Ma is the Hardy-Littlewood maximal sequence of $\mathbf{a}$.

Therefore,

$$T_{\nu}^{*} a(n) \le T_{\phi}^{*} a(n) + Ma(n).$$

Inequality (5.10) now follows from 3.6 and 2.4. For the proof of (5.11), fix N and consider

$$\sum_{|n-j|\le 4N} |\phi(n-j)|\ \psi\left(\frac{|n-j|}{N}\right) |a(j)| \le \sum_{4N\ge|n-j|>N/2} |\phi(n-j)|\ |a(j)|$$

$$\le C_2 \sum_{4N\ge|n-j|>N/2} \frac{|a(j)|}{|n-j|}$$

$$\le \frac{C}{8N+1} \sum_{|n-j|\le 4N} |a(j)|$$

$$\le C\, Ma(n).$$

Therefore

$$T_{|\psi|}^{*} a(n) \le C\, Ma(n).$$

It follows that $T_{|\psi|}^{*}$ is bounded on ℓ^{p}.

Hence the proof of Lemma 5.9 is complete.

In the following Proposition we prove the boundedness of the operators $[\mathbf{b},T_\nu]^*$ and $[\mathbf{b},T_{|\psi|}]^*$ on the ℓ^p spaces.

5.12. Proposition. Let $1 < p < \infty$ and $\mathbf{b} \in$ BMO(Z). Then there exists a constant $C_p > 0$ such that

$$\|[\mathbf{b},\ T_\nu]^*\ \mathbf{a}\|_p \le C_p\ \|\mathbf{a}\|_p,\ \forall\ \mathbf{a} \in \ell^p \tag{5.13}$$

$$\|[\mathbf{b},\ T_{|\psi|}]^*\ \mathbf{a}\|_p \le C_p\ \|\mathbf{a}\|_p,\ \forall\ \mathbf{a} \in \ell^p \tag{5.14}$$

Proof. For $J = 1,2,3,\dots,$ define

$$V_J\ \mathbf{a}(n) = \sup_{N\le J} \left|\sum_{j=n-N}^{n+N} [b(n)-b(j)]\ \nu_N(n-j)a(j)\right|$$

and $$W_J\ \mathbf{a}(n) = \sup_{N\le J} \left|\sum_{j=n-4N}^{n+4N} |b(n)-b(j)|\ |\psi_N(n-j)|\ |a(j)|\right|.$$

Then $$[\mathbf{b},T_\nu]^*\ \mathbf{a}(n) = \sup_J V_J\mathbf{a}(n)$$

and $$[\mathbf{b},T_{|\psi|}]^*\ \mathbf{a}(n) = \sup_J W_J\mathbf{a}(n).$$

We will prove below that

$$\|V_J\mathbf{a}\|_p \le C\|\mathbf{a}\|_p,\ \forall\ \mathbf{a} \in \ell^p \tag{5.15}$$

$$\|W_J\mathbf{a}\|_p \le C'\ \|\mathbf{a}\|_p,\ \forall\ \mathbf{a} \in \ell^p \tag{5.16}$$

where the constants C and C' are independent of $\mathbf{a}$ and J. Then the proposition follows by monotone convergence theorem.

For the proof of inequalities (5.15) and (5.16), we first obtain estimates for the corresponding sharp maximal sequences. Then we will prove Proposition 5.12 using Theorem 4.7 These estimates are proved in the following lemma.

5.17. Lemma. Let $r > 1$ and $\mathbf{a} = \{a(n)\}$ be a sequence. Then there exist constants C and $C' > 0$ such that

$$\sup_J (V_J\mathbf{a})^{\#}(n) \le C\|\mathbf{b}\|_* \left\{\left[M(T_\nu^*\mathbf{a})^r(n)\right]^{1/r} + \left[M(|\mathbf{a}|^r)(n)\right]^{1/r}\right\} \qquad (5.18)$$

and

$$\sup_J (W_J\mathbf{a})^{\#}(n) \le C'\|\mathbf{b}\|_* \left\{\left[M(T_{|\psi|}^*\mathbf{a})^r(n)\right]^{1/r} + \left[M(|\mathbf{a}|^r)(n)\right]^{1/r}\right\} \qquad (5.19)$$

Proof. (i) Proof of inequality (5.18) : Fix $J \ge 1$ and $n \in Z$. Then if I is any interval containing n, put

$$C_I = \sup_{N\le J} \left|\sum_{i=-\infty}^{\infty} [b(i) - b_I]\ \nu_N(j_o-i)\ a\chi_{Z\setminus 2I}(i)\right|,$$

where j_o is the centre of I. Then, for $j \in I$,

$$|V_J\mathbf{a}(j) - C_I| = \left|\sup_{N\le J}\left|\sum_{i=-\infty}^{\infty} [b(j)-b(i)]\ \nu_N(j-i)\ a(i)\right| - \sup_{N\le J}\left|\sum_{i=-\infty}^{\infty} [b_I-b(i)]\ \nu_N(j_o-i)\ a\chi_{Z\setminus 2I}(i)\right|\right|$$

$$\le \sup_{N\le J}\left|\sum_{i=-\infty}^{\infty} [b(j)-b(i)]\ \nu_N(j-i)\ a(i) - \sum_{i=-\infty}^{\infty} [b_I-b(i)]\ \nu_N(j_o-i)\ a\chi_{Z\setminus 2I}(i)\right|$$

$$\le \sup_{N\le J}\left|\sum_{i=-\infty}^{\infty} [b(j)-b_I]\ \nu_N(j-i)\ a(i)\right|$$

$$+ \sup_{N\le N}\left|\sum_{i=-\infty}^{\infty} [b(i)-b_I]\ \nu_N(j_o-i)\ a\chi_{2I}(i)\right|$$

$$+ \sup_{N\le N}\left|\sum_{i=-\infty}^{\infty} [b(i)-b_I]\ [\nu_N(j-i) - \nu_N(j_o-i)]a\chi_{Z\setminus 2I}(i)\right|$$

$$\equiv A_1(j) + A_2(j) + A_3(j).$$

For the first term, we have, with $\frac{1}{r} + \frac{1}{r'} = 1$,

$$\frac{1}{\text{card } I} \sum_{j\in I} A_1(j) \le \frac{1}{\text{card } I} \sum_{j\in I} |b(j)-b_I|\ T_\nu^*\mathbf{a}(j)$$

$$\le \left(\frac{1}{\text{card } I} \sum_{j\in I} |b(j) - b_I|^{r'}\right)^{1/r'} \left(\frac{1}{\text{Card } I} \sum_{j\in I} |T_\nu^*\mathbf{a}(j)|^r\right)^{1/r}$$

$$\le \|\mathbf{b}\|_* \, [M(T_\nu^*\mathbf{a})^r(n)]^{1/r}, \text{ using Cor. 4.5.}$$

Now consider

$$\frac{1}{\text{card } I} \sum_{j\in I} A_2(j) \le \frac{1}{\text{card } I} \sum_{j\in I} |T_\nu^*[(b-b_I)\mathbf{a}\chi_{2I}](j)|$$

$$\le \left(\frac{1}{\text{card } I} \sum_{j\in I} |T_\nu^*[(b - b_I)\ \mathbf{a}\chi_{2I}](j)|^s\right)^{1/s}$$

where $s > 1$. We can further replace the above summation over I by a summation over Z. Then using the boundedness of T_ν^* on ℓ^s (Lemma 5.8) we get

$$\frac{1}{\text{card } I} \sum_{j\in I} A_2(j) \le C\left(\frac{1}{\text{card } I} \sum_{j\in 2I} |b(j)-b_I|^s\ |a(j)|^s\right)^{1/s}$$

$$\le C\left(\frac{1}{\text{card } 2I} \sum_{j\in 2I} |b(j) - b_I|^{sq}\right)^{1/sq} \left(\frac{1}{\text{Card } 2I} \sum_{j\in 2I} |a(j)|^{sq'}\right)^{1/sq'},$$

where $q > 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Now

$$\left(\frac{1}{\text{card } 2I} \sum_{j\in 2I} |b(j) - b_I|^{sq}\right)^{1/sq}$$

$$\le C\left(\frac{1}{\text{card } 2I} \sum_{j\in 2I} |b(j) - b_{2I}|^{sq}\right)^{1/sq} + |b_{2I} - b_I|$$

$$\le C\|\mathbf{b}\|_* \text{ . [by 4.5 and 4.6]}$$

Therefore

$$\frac{1}{\text{card } I} \sum_{j\in I} A_2(j) \le C\|\mathbf{b}\|_* \, [M(|\mathbf{a}|^r)(n)]^{1/r}$$

provided we choose s and q′ so that $sq' = r$. It remains to estimate $A_3(j)$, for $j \in I$. We have

$$A_3(j) \le \sup_{N\le J} \sum_{i=-\infty}^{\infty} |b(i)-b_I| \, |\nu_N(j-i)-\nu_N(j_o-i)| \, |a\chi_{Z\setminus 2I}(i)|$$

But since $i \notin 2I$ and $j \in I$, $|j_o-i| > 2|j-j_o|$.

Therefore

$$|\nu_N(j-i) - \nu_N(j_o-i)| \le \frac{C|j-j_o|}{(j-i)^2} \le \frac{C|j-j_o|}{(j_o-i)^2} .$$

Therefore

$$A_3(j) \le C \sup_{N\le J} \sum_{i=-\infty}^{\infty} |b(i)-b_I| \frac{|j-j_o|}{(j_o-i)^2} |a\chi_{Z\setminus 2I}(i)|$$

$$\le C(\text{card } I) \left(\sum_{i\notin 2I} \frac{|b(i)-b_I|^{r'}}{|j_o-i|^{r'}}\right)^{1/r'} \left(\sum_{i\notin 2I} \frac{|a(i)|^r}{|j_o-i|^r}\right)^{1/r} .$$

Now by [4.6],

$$\left(\sum_{i\notin 2I} \frac{|b(i)-b_I|^{r'}}{|j_o-i|^{r'}}\right)^{1/r'} \le \frac{C\|b\|_*}{(\text{card } I)^{1/r}} .$$

Now let $I_k = 2^k I$,

$$\left(\sum_{i\notin 2I} \frac{|a(i)|^r}{|j_o-i|^r}\right)^{1/r} = \left(\sum_{k=1}^{\infty} \sum_{i\in I_{k+1}\setminus I_k} \frac{|a(i)|^r}{|j_o-i|^r}\right)^{1/r}$$

$$\le \left(\sum_{k=1}^{\infty} \sum_{i\in I_{k+1}\setminus I_k} \frac{|a(i)|^r}{2^{kr}(\text{card } I)^r}\right)^{1/r}$$

$$\leq \left[\sum_{k=1}^{\infty} \frac{2}{2^{k(r-1)}(\text{card } I)^{r-1}} \frac{1}{2^{k+1}(\text{card } I)} \sum_{i\in I_k} |a(i)|^r\right]^{1/r}$$

$$\leq \frac{1}{(\text{card } I)^{1/r'}} \left[\sum_{k=1}^{\infty} \frac{2}{2^{k(r-1)}} M(|\mathbf{a}|^r)(n)\right]^{1/r}$$

$$\leq \frac{C[M(|a|^r)(n)]^{1/r}}{(\text{card } I)^{1/r'}} .$$

Therefore , $A_3(j) \leq C \; \|\mathbf{b}\|_* \; [M(|\mathbf{a}|^r)(n)]^{1/r}$

and so

$$\frac{1}{\text{card } I} \sum_{j\in I} A_3(j) \leq C\|\mathbf{b}\|_* \; [M(|\mathbf{a}|^r)(n)]^{1/r}.$$

Hence inequality (5.18) is proved.

(ii) <u>Proof of inequality (5.19)</u> : For $n \in Z$ and any interval I containing n, we choose

$$C_I = \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(i) - b_I| \quad |\psi_N(j_o-i)| \quad |a\chi_{Z\backslash 2I}(i)|,$$

where j_o is the centre of I.

Then $|W_J\mathbf{a}(j) - C_I|$

$$= \left| \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(j)-b(i)| \quad |\psi_N(j-i)| \quad |a(i)| \right.$$

$$\left. - \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(i)-b_I| \quad |\psi_N(j_o-i)| \quad |a\chi_{Z-2I}(i)| \right|$$

$$\leq \sup_{N\leq J} \sum_{i=-\infty}^{\infty} \Big| |b(j)-b(i)| \quad |\psi_N(j-i)| \quad |a(i)|$$

$$- \quad |b(i)-b_I| \quad |\psi_N(j_o-i)| \quad |a\chi_{Z-2I}(i)| \Big|$$

$$\leq \sup_{N\leq J} \sum_{i=-\infty}^{\infty} \Big|\Big|[(b(j)-b_I) \ |\psi_N(j-i)| \ |a(i)|$$

$$+ (b_I - b(i)) \ |\psi_N(j-i)| \ |a\chi_{2I}(i)|]$$

$$+ (b_I - b(i)) \ |\psi_N(j-i)| \ |a\chi_{Z\setminus 2I}(i)|\Big|$$

$$- |b_I-b(i)| \ |\psi_N(j_o-i)| \ |a\chi_{Z\setminus 2I}(i)|\Big| \ .$$

Since $\big||x+y| - |z|\big| \leq |x| + \big||y| - |z|\big|$, $\forall\ x,y,z \in \mathbb{C}$,

we have

$$|W_Ja(j) - C_I|$$

$$\leq \sup_{N\leq J} \sum_{i=-\infty}^{\infty} \Big|(b(j)-b_I) \ |\psi_N(j-i)| \ |a(i)|$$

$$+ (b_I - b(i)) \ |\psi_N(j-i)| \ |a\chi_{2I}(i)|\Big|$$

$$+ \Big||(b(i) - b_I| \ |\psi_N(j-i)| \ |a\chi_{Z\setminus 2I}(i)|$$

$$- |b(i)-b_I| \ |\psi_N(j_o-i)| \ |a\chi_{Z-2I}(i)|\Big|$$

$$\leq \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |[b(j)-b_I| \ |\psi_N(j-i)| \ |a(i)|$$

$$+ \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(i) - b_I| \ |\psi_N(j-i)| \ |a\chi_{2I}(i)|$$

$$+ \sup_{N\leq J} \sum_{i=-\infty}^{\infty} |b(i) - b_I| \ |\psi_N(j-i)-\psi_N(j_o-i)| \ |a\chi_{Z\setminus 2I}(i)|$$

$$\equiv B_1(j) + B_2(j) + B_3(j).$$

The estimates for each of these terms are obtained exactly as in the previous case by replacing ν by ψ.

This completes the proof of Lemma 5.17.

5.20 Proof of Proposition 5.12.

We can now complete the proof of inequalities (5.15) and (5.16).

Choose $r < p$. Then

$$\|(V_J\mathbf{a})^{\#}\|_p = \left(\sum_{n=-\infty}^{\infty} |(V_J\mathbf{a})^{\#}(n)|^p \right)^{1/p}$$

$$\leq C \left[\left(\sum_{n=-\infty}^{\infty} [M(T_\nu^*\mathbf{a})^r(n)]^{p/r} \right)^{1/p} + \left(\sum_{n=-\infty}^{\infty} [M(|\mathbf{a}|^r)(n)]^{p/r} \right)^{1/p} \right]$$

$$\leq C\left[\left(\sum_{n=-\infty}^{\infty} [T_\nu^*\mathbf{a}(n)]^p \right)^{1/p} + \left(\sum_{n=-\infty}^{\infty} |a(n)|^p \right)^{1/p} \right]$$

$$\leq C\|\mathbf{a}\|_p \ . \ [\text{by } (5.10)].$$

Similarly, we can prove that $\|(W_J\mathbf{a})^{\#}\|_p \leq C'\|\mathbf{a}\|_p$.

It remains to prove that $V_J\mathbf{a}$ and $W_J\mathbf{a}$ belong to ℓ^p. Then inequalities (5.15) and (5.16) hold by the sharp maximal sequence theorem. The ℓ^p norms of $V_J\mathbf{a}$ and $W_J\mathbf{a}$ may depend on J (see Remark 4.8). We claim that

$$V_J\mathbf{a}(n) \leq C_J \|\mathbf{b}\|_* (M(|\mathbf{a}|^r)(n))^{1/r}, \ 1 < r < \infty$$

and

$$W_J\mathbf{a}(n) \leq C_J\|\mathbf{b}\|_* T_{|\psi|}^*\mathbf{a}(n).$$

We have

$$V_J\mathbf{a}(n) = \sup_{N\leq J} \left| \sum_{i=n-N}^{n+N} [b(n)-b(i)] \ \nu_N(n-i)a(i) \right|$$

$$= \sup_{N\leq J} \left| \sum_{i=-\infty}^{\infty} [b(n)-b(i)] \ \nu_N(n-i)a\chi_{I_J}(i) \right|.$$

where $I_J = [n-J, n+J]$. We estimate this exactly as we estimated the term A_2 in Lemma 5.17 and we have

$$V_J\mathbf{a}(n) \leq (2J+1)\left[C\|\mathbf{b}\|_* + (\log J)\ \|\mathbf{b}\|_*\right]\left(M(|\mathbf{a}|^r)(n)\right)^{1/r}$$

$$\leq C_J\ \|\mathbf{b}\|_*\ \left(M(|\mathbf{a}|^r)(n)\right)^{1/r}.$$

Therefore choosing $r < p$ we see that $V_J\mathbf{a} \in \ell^p$ for $\mathbf{a} \in \ell^p$.

Next let $n \in Z$ and $I_{4J} = [n-4J,\ n+4J]$,

Then for $i \in I_{4J}$

$$|b(n) - b(i)| \leq |b(n) - b_{I_{4J}}| + |b_{I_{4J}} - b(i)|$$

$$\leq 2(8J+1)\ \|\mathbf{b}\|_*.$$

Therefore

$$W_J\mathbf{a}(n) = \sup_{N\leq J} \sum_{i=n-4N}^{n+4N} |b(n)-b(i)|\ \psi_N(n-i)|\ |a(i)|$$

$$\leq C_J\ \|\mathbf{b}\|_*\ T^*_{|\psi|}\ a(n).$$

Hence $W_J\mathbf{a} \in \ell^p$, $\forall\ \mathbf{a} \in \ell^p$ by (5.11).

This completes the proof of Proposition 5.12.

5.21. Corollary. Let $1 < p < \infty$. The commutator of the S-operator $[\mathbf{b},T_\phi]\mathbf{a}$ exists for every $\mathbf{a} \in \ell^p$.

Proof. Note that finite sequences are dense in ℓ^p and $[\mathbf{b},T_\phi]\mathbf{a}$ exists for every finite sequence $\mathbf{a}$.. Since we have already proved $[\mathbf{b},T_\phi]^*$ is bounded on ℓ^p, the proof follows at once by theorem 2.5.

2 Commutators of ergodic S-Operators

5.22. Let $(X,\mathbb{B},m)$ be a probability space and U an invertible measure preserving transformation on X. In this section we study the commutator of the operator of pointwise multiplication by a function $\mathbf{b} \in L^1(X)$ and an S-operator. This is defined as

$$[\mathbf{b},\tilde{T}_\phi]\, f(x) = b(x)\, \tilde{T}_\phi\, f(x) - \tilde{T}_\phi(\mathbf{b}f)(x)$$

$$= \sum_{k=-\infty}^{\infty} \phi(k)\, [b(x) - b(U^{-k}x)]\, f(U^{-k}x).$$

We first show that this operator is well-defined on a dense subset of $L^p(X)$, $1 \le p \le 2$.

<u>**Lemma.**</u> Let $1 \le p \le 2$ and $D = \{f \in L^p(X) : f = f \circ U \text{ a.e.}\}$

$$\cup\{f \in L^p(X) : f = g-g \circ U \text{ a.e.},\ g \in L^\infty(X)\}.$$

Then for each f in D, $[\mathbf{b},\tilde{T}_\phi]f$ exists a.e.

<u>**Proof.**</u> If $f \in L^p(X)$ with $f = f \circ U$ a.e., then

$$[\mathbf{b},\tilde{T}_\phi]\, f(x) = b(x)\, \tilde{T}_\phi\, f(x) - f(x)\, \tilde{T}_\phi b(x),$$

which exists a.e. since $\mathbf{b} \in L^1(X)$.

If $f = g-g\circ U$, $g \in L^\infty(X)$, then $\mathbf{b}.f \in L^1(X)$, so that

$$[\mathbf{b},\tilde{T}_\phi]\, f(x) = b(x)\, \tilde{T}_\phi f(x) - \tilde{T}_\phi(b.f)(x),$$

which exists a.e. Hence the proof of the lemma is complete.

For a.e. $x \in X$, let $f_x(n) = f(U^n x)$. For $K \in Z_+$, define

$$f_x^K(n) = \begin{cases} f_x(n) & \text{if } |n| \le K \\ 0 & \text{if } |n| > K. \end{cases}$$

Since $[\mathbf{b}_x, T_\phi]_J^*$ is sub linear, for $K, L \in Z_+$ and $n \in Z$, we have

$$[\mathbf{b}_x, T_\phi]_J^* f_x(n) \le [\mathbf{b}_x, T_\phi]_J^* f_x^{K+L}(n) + [\mathbf{b}_x, T_\phi]_J^* (f_x - f_x^{K+L})(n)$$

We can choose L large enough so that

$$\operatorname{supp} [\mathbf{b}_x, T_\phi]_J^* (f_x - f_x^{K+L}) \subseteq \{n: |n| > K\},$$

since $\operatorname{supp} (f_x - f_x^{K+L}) \subseteq \{n: |n| > K+L\}$.

Note that L depends only on J and not on K.

Therefore

$$[\mathbf{b}_x, T_\phi]_J^* f_x(n) \le [\mathbf{b}_x, T_\phi]_J^* f_x^{K+L}(n) \quad \text{for } |n| \le K.$$

Also for a.e. $x \in X$ and $j \in Z$, we have

$$\begin{aligned} [\mathbf{b}, \tilde{T}_\phi]_J^* f(U^j x) &= \sup_{N \le J} \Big| \sum_{k=-N}^{N} \phi(k) \, [b(U^j x) - b(U^{j-k} x)] f(U^{j-k} x) \Big| \\ &= \sup_{N \le J} \Big| \sum_{k=-N}^{N} \phi(k) \, [b_x(j) - b_x(j-k)] \, f_x(j-k) \Big| \\ &= [\mathbf{b}_x, T_\phi]_J^* f_x(j). \end{aligned}$$

Then

$$\int_X ([\mathbf{b},\tilde{T}_\phi]_J^* f(x))^p \, dm = \frac{1}{2K+1} \sum_{j=-K}^{K} \int_X ([\mathbf{b},\tilde{T}_\phi]_J^* f(U^j x))^p \, dm$$

$$= \frac{1}{2K+1} \sum_{j=-K}^{K} \int_X ([\mathbf{b}_x,T_\phi]_J^* f_x(j))^p \, dm$$

$$\leq \frac{1}{2K+1} \int_X \sum_{j=-K}^{K} ([\mathbf{b}_x,T_\phi]_J^* f_x^{K+L}(j))^p \, dm$$

$$\leq \frac{C}{2K+1} \int_X \sum_{j=-(K+L)}^{(K+L)} |f_x(j)|^p \, dm$$

$$= \frac{C}{2K+1} \sum_{j=-(K+L)}^{(K+L)} \int_X |f(U^j x)|^p \, dm$$

$$\leq \frac{C[2(K+L)+1]}{2K+1} \cdot \|f\|_p^p.$$

Choosing K sufficiently large, we get

$$\|[\mathbf{b},\tilde{T}_\phi]_J^* f\|_p \leq C_p \|f\|_p.$$

Hence the proof of Theorem 5.23 is complete.

5.24. **Corollary.** Let $1 < p < \infty$ and $\mathbf{b} \in BMO(X)$. Then the commutator of the ergodic S-operator $[\mathbf{b},\tilde{T}_\phi]f$ exists a.e., $\forall\ f \in L^p(X)$.

Proof. For $1 \leq p \leq 2$, The set $D = \{f \in L^p(X) : f = f \circ U \text{ a.e.}\}$

$$\cup \{f \in L^p(X) : f = g - g \circ U \text{ a.e.}, g \in L^\infty(X)\}$$

is dense in $L^p(X)$. Since we have already proved the a.e. existence of $[\mathbf{b},\tilde{T}_\phi]f$, for each $f \in D$, the a.e. existence of

$[\mathbf{b},\tilde{T}_\phi]f$, for each $f \in L^p(X)$, follows from Theorem 5.23 and 2.5.

Since $(X,\mathbb{B},m)$ is a probability space, $L^p(X) \subseteq L^2(X)$ for $p > 2$, we have the a.e. existence of $[\mathbf{b},\tilde{T}_\phi]f$, $\forall\, f \in L^p(X)$, $1 < p < \infty$.

In the next theorem we prove the converse of Theorem 5.23, when $\tilde{T}_\phi$ is the ergodic Hilbert transform.

5.25. Theorem. Let $(X,\mathbb{B},m)$ be a non-atomic probability space, U an invertible, ergodic, measure preserving transformation on X and $\mathbf{b} \in L^1(X)$. If $[\mathbf{b},\tilde{H}]^*$ is bounded on $L^p(X)$, for some p, $1 < p < \infty$, then $\mathbf{b} \in BMO(X)$.

Proof. First note that the hypothesis implies that for each $N \in Z_+$ and $f \in L^p(X)$,

$$\int_X \Big|\sum_{k=-N}^{N}{}' \frac{[b(x)-b(U^{-k}x)]}{k} f(U^{-k}x)\Big|^p \, dm \le C\|f\|_p^p,$$

where the prime in the summation denotes the exclusion of the term $k=0$.

Let $\{a(k)\}$ be any sequence in ℓ^p. We will prove that

$$\sup_N \sum_{j=-N}^{N} \Big|\sum_{k=-N}^{N}{}' \frac{b_x(j)-b(k)]a(k)}{k}\Big|^p \le C\|\mathbf{a}\|^p, \text{ for a.e. } x \in X, \quad (5.26)$$

where for a.e. x, the sequence $\mathbf{b}_x$ is defined as $b_x(k) = b(U^k x)$.

If inequality (5.26) is proved, then by the same argument as in the proof of Theorem 5.3, it follows that $\mathbf{b}_x \in BMO(Z)$ a.e. x and $\|\mathbf{b}_x\|_* \le C$, where C is independent of x so that $\mathbf{b} \in BMO(X)$.

For inequality 5.26, fix $N \in Z_+$ and take an ergodic rectangle [see 2.9] $R = \bigcup_{j=-4N}^{4N} U^j E$, of length 8N+1 with base E. Let F be any measurable subset of E. Then F is also the base of an ergodic rectangle of length 8N+1.

Note that if $-N \le j \le N$ and if f is a function whose support is in the rectangle $R' = \bigcup_{k=-N}^{N} U^k F$, then for $x \in F$, we have

$$\sum_{k=\ 2N}^{2N,} \frac{[b(U^j x) - b(U^{j+k} x)]\ f(U^{j+k} x)}{k}$$

$$= \sum_{k\ =-N}^{N,} \frac{[b(U^j x) - b(U^k x)]\ f(U^k x)}{(k-j)} .$$

Now given a sequence $\{a(k)\} \in \ell^p$, define

$$f(U^k x) = \begin{cases} a(k) & \text{if } x \in F \text{ and } -N \le k \le N \\ 0 & \text{otherwisie.} \end{cases}$$

Since $[\mathbf{b}, \tilde{H}]^*$ is bounded on $L^p(X)$, we have

$$C\|f\|_p^p \ \ge \int_X |[\mathbf{b}, \tilde{H}]^*\ f(x)|^p\ dm$$

$$\ge \int_{R'} |\sum_{k=\ 2N}^{2N,} \frac{[b(x)-b(U^{-k}x)]}{k}\ f(U^{-k}x)|^p\ dm$$

$$\ge \int_{R'} |\sum_{k=-2N}^{2N,} \frac{[b(x)-b(U^{k}x)]}{k}\ f(U^{k}x)|^p\ dm$$

$$\geq \sum_{j=-N}^{N} \int_{U^jF} \left| \sum_{k=-2N}^{2N}{}' \frac{[b(x)-b(U^kx)]f(U^kx)}{k} \right|^p dm$$

$$= \sum_{j=-N}^{N} \int_{F} \left| \sum_{k=\ 2N}^{2N}{}' \frac{[b(U^jx)-b(U^{k+j}x)]f(U^{k+j}x)}{k} \right|^p dm$$

$$= \sum_{j=-N}^{N} \int_{F} \left| \sum_{k=-N}^{N}{}' \frac{[b(U^jx)-b(U^kx)]f(U^kx)}{(k-j)} \right|^p dm$$

$$= \int_{F} \sum_{j=-\ N}^{N} \left| \sum_{k=-\ N}^{N}{}' \frac{[b(U^jx)-b(U^kx)]f(U^{kj}x)}{k} \right|^p dm$$

But, $\|f\|_p^p = \int_{R'} |f(x)|^p \, dm$

$$= \sum_{k=-N}^{N} \int_{U^kF} |f(x)|^p \, dm$$

$$= \sum_{k=-N}^{N} \int_{F} |f(U^kx)|^p \, dm$$

$$= \left(\sum_{k=-N}^{N} |a(k)|^p \right) m(F).$$

Therefore

$$\frac{1}{m(F)} \int_{F} \sum_{j=-N}^{N} \left| \sum_{k=-N}^{N}{}' \frac{[b(U^jx) - b(U^kx)]}{(k-j)} a(k) \right|^p dm \leq C\|\mathbf{a}\|_p^p.$$

Sine F was an arbitrary subset of E, we get

$$\sum_{j=-N}^{N} \left| \sum_{k=-N}^{N}{}' \frac{[b(U^jx)-b(U^kx)]a(k)}{(k-j)} \right|^p \leq C\|\mathbf{a}\|_p^p, \text{ a.e. } x \in E.$$

Now since we have assumed that U is ergodic, X can be written as a countable union of bases of ergodic rectangles of length 8N+1 [see 2.9].

Therefore

$$\sum_{j=-N}^{N} \left| \sum_{k=-N}^{N,} \frac{[b(U^j x)-b(U^k x)]a(k)}{(k-j)} \right|^p \le C\|\mathbf{a}\|_p^p, \text{ a.e. } x \in X.$$

Since N was arbitrary we have

$$\sup_N \sum_{j=-N}^{N} \left| \sum_{k=-N}^{N,} \frac{[b(U^j x)-b(U^k x)]a(k)}{(k-j)} \right|^p \le C\|\mathbf{a}\|_p^p, \text{ a.e. } x \in X.$$

This proves inequality (5.26).

Hence the proof of Theorem 5.25 is complete.

5.26. CONCLUDING REMARKS.

1. In Theorem 5.25, we have assumed that (a) U is ergodic and (b) $[\mathbf{b},H]^*$ is a bounded operator on $L^p(X)$, $1 < p < \infty$. We have not been able to deduce that $\mathbf{b} \in BMO(X)$ under the hypothesis that $[\mathbf{b},H]$ is a bounded operator on $L^p(X)$, $1 < p < \infty$. Also, we do not know if the hypothesis of ergodicity of U can be dropped.

2. We have not considered the case $p = 1$ for the commutator problem. For $\mathbb{R}$ and $\mathbb{T}$, we refer to the work of Janson, Peetre and Semmes [23]. In the discrete case of $\mathbb{Z}$, it is not hard to see from the proof in [23] that if $\mathbf{b} \in BMO(\mathbb{Z})$ and $[\mathbf{b},H]: h^1_{at} \to \ell^1$ is bounded, then $\mathbf{b}$ is a constant sequence. The commutator problem for the ergodic Hardy space $H^1(X)$ remains open, for a future investigation.

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